

# The Pachner graph of 2-spheres

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*It is well-known that the Pachner graph of triangulated  $n$ -vertex 2-spheres is connected, i.e., each pair of  $n$ -vertex 2-spheres can be turned into each other by a sequence of edge flips. In this article, we study various induced subgraphs of this graph. In particular, we prove that the subgraph of  $n$ -vertex flag 2-spheres distinct from the double cone is still connected. In contrast, we show that the subgraph of  $n$ -vertex stacked 2-spheres has at least as many connected components as there are trees on  $\lfloor \frac{n-5}{3} \rfloor$  nodes of maximum degree  $\leq 4$ .*

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## 1 Introduction

The *Pachner graph* is the graph, whose vertices are PL triangulations of a manifold, and two vertices are connected by an edge if and only if their corresponding triangulations can be transformed into each other by a single *bistellar move*, i.e., a combination of a stellar subdivision and an inverse of a stellar subdivision. It follows from a theorem by Pachner (Theorem 2.8, [13]), that two triangulations represent the same PL manifold, if and only if there is a sequence of bistellar moves, turning one into a triangulation isomorphic to the other. In other words, the Pachner graph of triangulations of a fixed PL  $d$ -manifold is connected.

Finding a sequence of bistellar moves between triangulations – or concluding that there is no such sequence – thus solves the PL homeomorphism problem. Consequently, studying the properties of the Pachner graph, even of a single fixed manifold, is of great importance. And being able to purposefully “navigate” in this space from one region to another is a highly desirable capability in the field of combinatorial topology.

In general, studying the Pachner graph has two major difficulties: (i) The Pachner graph is of infinite size, and (ii) the homeomorphism problem is undecidable in dimensions  $\geq 4$ .

The first difficulty can be partially overcome by observing that the Pachner graph has a natural graded structure into finite sets of  $n$ -vertex triangulations,  $n$  fixed<sup>1</sup>. Naturally, the arcs within a level correspond to all bistellar moves but stellar subdivisions of facets and their inverses (called 0- and  $d$ -moves). The arcs corresponding to 0- and  $d$ -moves connect different levels of the grading.

The second difficulty is inherent to the problem. Hence, we cannot expect to be able to extract any useful information about, for example, the diameter from the Pachner graph of a higher dimensional manifold – at least not in the general setting.

It follows that studying the Pachner graph seems most promising in dimensions less than four.

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<sup>1</sup>Note that these sets are not finite if more general types of triangulations are considered.

**In dimension 1**, the Pachner graph of the only closed connected 1-manifold, the circle, is nothing but an infinite path.

**In dimension 2** or, more specifically, in the case of the Pachner graph of the 2-sphere, it is known how to navigate from an arbitrary  $n$ -vertex 2-sphere down to the boundary of the tetrahedron using only  $O(n)$  bistellar moves. It is well-known that each level, i.e., the Pachner graph of all triangulated  $n$ -vertex 2-spheres,  $n \geq 4$  fixed, is connected [15] and of size bounded by a singly exponential function in  $n$ . Moreover, the length of a sequence of 1-moves, or *edge flips*, connecting two triangulated  $n$ -vertex 2-spheres is bounded above by  $5.2n - 24.4$  [3]. At the same time, there exist triangulations which are at least  $2n - 14$  edge flips away from the double cone with  $n$ -vertices, see Komuro [8]. Naturally, these bounds act as upper and lower bounds for the diameter of the Pachner graph of  $n$ -vertex 2-spheres. See Section 2.4 for further details, and [4] for a comprehensive survey on upper and lower bounds for the diameter of the Pachner graph of  $n$ -vertex 2-spheres.

**In dimension 3** much less is known. The best upper bound for distances in the Pachner graph is given by  $O(t^2 2^{ct^2})$  for the number of moves between a  $t$ -tetrahedron triangulation of  $S^3$  and the boundary of the 4-simplex, see Mijatović [11]<sup>2</sup>. Moreover, the  $n$ -th level of the Pachner graph is not connected anymore<sup>3</sup>. To see this, consider an  $n$ -vertex triangulation of the 3-sphere containing (i) no edge of degree 3 and (ii) the complete graph with  $n$  vertices as one-skeleton. Such a triangulation neither allows 1-moves (sometimes also called 2-3-moves), nor 2-moves (sometimes also called 3-2-moves), and is, thus, isolated in the Pachner graph of  $n$ -vertex 3-sphere triangulations. See [14] for a number of examples of such 3-sphere triangulations.

In this article, we focus on the Pachner graph of  $n$ -vertex triangulated 2-spheres. Since 0- and 2-moves (also called 1-3-moves and 3-1-moves) change the number of vertices, all edges of this graph represent edge flips (1-moves, also called 2-2-moves). More precisely, we want to look at induced subgraphs of this graph, defined by some prominent subfamilies of the set of  $n$ -vertex triangulations of the 2-sphere.

Namely, we focus on what is called *stacked* and *flag 2-spheres* (see Sections 2.2 and 2.3 for details). In many ways, flag 2-spheres are the counterpart to stacked 2-spheres. While stacked 2-spheres contain the maximum number of induced 3-cycles, flag 2-spheres do not contain any such cycles. Moreover, every triangulated 2-sphere can be decomposed into a collection of flag 2-spheres by iteratively cutting along its induced 3-cycles and pasting the missing triangles. For stacked 2-spheres this decomposition yields the maximum number of connected components, each isomorphic to the boundary complex of the tetrahedron.

In [12] the authors give upper bounds for the number of edge flips connecting two flag 2-spheres within the class of Hamiltonian triangulations of the 2-sphere, see Theorem 2.10. Our main result states that such a sequence of edge flips exists even *within the class of flag 2-spheres* – as long as both triangulations are distinct from the double cone  $\Gamma_n = S_{n-2}^1 * S_2^0$  over the  $(n - 2)$ -gon (Figure 3.4, left), see Theorem 3.1. Observe that the  $n$ -vertex double cone  $\Gamma_n$ ,  $n \geq 6$ , is a flag 2-sphere which cannot be connected to any other flag 2-sphere since every edge contains a degree four vertex. Thus every edge flip produces a degree 3 vertex and the resulting complex is not flag.

This theorem complements a result by Lutz and Nevo stating that every pair of  $d$ -dimensional flag complexes,  $d \geq 3$ , are connected by a sequence of edge subdivisions, and edge contractions [9].

In contrast, the subgraph of the Pachner graph of  $n$ -vertex stacked 2-spheres appears to have much less uniform properties. In particular, it is not connected in general. In Section 4 we give a precise condition on when exactly an edge flip of a stacked 2-sphere produces another stacked 2-sphere, see Theorem 4.1. Using this result, we then follow that the number of connected components of the Pachner graph of  $n$ -vertex stacked 2-spheres is exponential in  $n$ , see Corollary 4.7. Furthermore, we show that a pair of  $n$ -vertex stacked 2-spheres can be connected by a sequence of  $n$ -vertex stacked 2-spheres, each related to the previous one by an edge flip, if their associated stacked 3-balls have a dual graph without degree four vertices, see Theorem 4.9. These results are complemented by additional experimental data on the subgraph for  $n \leq 14$  vertices, see Table 1.

<sup>2</sup>This bound takes into account not just simplicial complexes but also generalised triangulations which, potentially, provide shortcuts in the Pachner graph.

<sup>3</sup>For generalised triangulations the levels of the Pachner graph are connected [10].

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## 2 Preliminaries

### 2.1 Abstract simplicial complexes and PL triangulations of manifolds

All triangulations in this article are described by abstract simplicial complexes. An abstract simplicial complex  $C$  is a collection of subsets of a finite set  $V(C)$ , called the *set of vertices* of  $C$ , such that every subset of an element is also an element. For  $i \geq 0$ , the elements of size  $i + 1$  are called the *i-simplices* (or *i-faces* or *faces of dimension i*) of  $C$ . The empty set  $\emptyset$  is a face (of dimension  $-1$ ) for every simplicial complex. An *i-face*  $\{v_0, \dots, v_i\}$  is also denoted by  $v_0v_1 \dots v_i$ . The *dimension* of  $C$  is the size of the largest element of  $C$  minus one.

Two simplicial complexes  $C$  and  $D$  are called *combinatorially isomorphic* (or sometimes just *isomorphic*) if there exists a bijection  $f : V(C) \rightarrow V(D)$  such that  $\alpha \in C \Leftrightarrow f(\alpha) \in D$ . A simplicial complex  $D$  is said to be a subcomplex of a simplicial complex  $C$  if  $D \subseteq C$ .

For an  $i$ -dimensional simplicial complex  $C$  and a  $j$ -dimensional complex  $D$  with disjoint vertex sets, their *simplicial join* (or sometimes just *join*) is the  $(i + j + 1)$ -dimensional simplicial complex

$$A * B = \{\delta \cup \gamma \mid \delta \in A, \gamma \in B\}.$$

A simplicial complex  $C$  is called *pure* if all its maximal simplices (that is, faces of maximal size, called *facets*) have the same dimension. For a face  $\delta \in C$ , the subcomplex  $\text{st}_C(\delta) = \{\Delta \in C \mid \delta \subset \Delta\}$  of all faces containing  $\delta$  is called the *star of  $\delta$  in  $C$* . The complex  $\text{lk}_C(v) = \{\Delta \in \text{st}_C(\delta) \mid \delta \cap \Delta = \emptyset\}$  of faces of the star of  $\delta$  disjoint to  $\delta$  is called the *link of  $\delta$  in  $C$* . The set of all  $i$ -dimensional faces of  $C$  is called the *i-skeleton*  $\text{Skel}_i(C)$ . The vector  $f(C) = (f_0, f_1, \dots, f_d)$ , where  $f_i = |\text{Skel}_i(C)|$  denotes the number of  $i$ -dimensional faces, is called the *f-vector* of  $C$ . Given an  $i$ -face  $\delta$  of  $C$ , the number of  $(i + 1)$ -faces containing  $\delta$  is referred to as its *degree*, written  $\deg_C(\delta)$ .

A  $d$ -dimensional pure simplicial complex is said to be a weak pseudomanifold, if every codimension one face is contained in at most two facets. By the boundary  $\partial C$  of a weak pseudomanifold  $C$  we mean the simplicial complex defined by the codimension one faces of  $C$  contained in only one facet. For a weak pseudomanifold  $C$ , its *dual graph*  $\Lambda(C)$  is the graph whose nodes correspond to the facets of  $C$  and two nodes are connected by an arc if and only if their facets share a common codimension one face. Note that, whenever we talk about the dual graph, we refer to its vertices as *nodes* and to its edges as *arcs* to avoid confusion with the vertices and edges of a simplicial complex.

An abstract simplicial complex  $C$  is said to be a *piecewise linear (PL) triangulation of a  $d$ -manifold*, or *combinatorial  $d$ -manifold*, if all of its vertex links are triangulated  $(d - 1)$ -spheres with piecewise linear structure. A *triangulated 2-sphere* (a *triangulated 3-ball*, a *triangulated PL manifold  $M$* ) is a PL triangulation of a 2-manifold (3-manifold,  $d$ -manifold) with underlying set piecewise linear homeomorphic to  $S^2$  ( $B^3$ ,  $M$  respectively). The simplest triangulated 2-sphere is the boundary complex of the tetrahedron, called the *standard 2-sphere*. The simplest triangulated 3-ball is the tetrahedron itself together with all of its faces, called the *standard 3-ball*. Triangulated 2-spheres are sometimes defined by so-called *planar triangulations*, i.e., finite simple planar graphs with  $n$  vertices containing the maximum number of  $3n - 6$  edges. See [2] for a standard reference containing further terminology for graphs.

Given a simplicial complex  $C$  and a subset  $W \subset V(C)$  of its set of vertices, the *subcomplex induced by  $W$* , denoted  $C[W]$  is the simplicial complex of all faces of  $C$  with vertices in  $W$ .

For the purpose of this article we consider graphs to be one-dimensional abstract simplicial complexes, and apply simplicial complex terminology to graphs where possible.

## 2.2 Flag and Hamiltonian 2-spheres

There are several special types of triangulated 2-spheres which are relevant for this article. The most important ones are introduced in this section and in Section 2.3.

**Definition 2.1** (Flag 2-sphere). A *flag 2-sphere* is a triangulated 2-sphere in which every cycle of three edges bounds a triangle.

Every triangulated 2-sphere  $S$  can be decomposed into flag spheres. Simply cut along a 3-cycle not bounding a triangle, and fill in the missing triangle in both parts. Iterating this procedure results in a set of flag spheres, called the primitive components of  $S$ . Identifying each one of them by a node, and the 3-cycles by arcs between nodes this defines a tree. If the tree is a single vertex,  $S$  is called *primitive*. Furthermore, a triangulated 2-sphere is called *4-connected* if its 1-skeleton is a 4-connected graph. Naturally, we have that a 2-sphere is 4-connected if and only if it is primitive if and only if it is flag.

**Definition 2.2** (Hamiltonian 2-sphere). A *Hamiltonian 2-sphere* is a triangulated 2-sphere containing a Hamiltonian cycle in its one-skeleton.

Hamiltonian 2-spheres play an important role in the proofs of upper bounds for the diameter of the Pachner graph of  $n$ -vertex 2-spheres, see [4] for an overview. Moreover, it is known that the Pachner graph of Hamiltonian 2-spheres is connected and has small diameter [12]. A flag 2-sphere is necessarily Hamiltonian [16], but the converse is not true.

## 2.3 Stacked balls and stacked spheres

Let  $\bar{\sigma}$  be the *standard  $d$ -ball*, i.e., a single  $d$ -simplex  $\sigma$  together with its faces. A simplicial complex  $C$  is called a *stacked  $d$ -ball* if there exists a sequence  $B_1, \dots, B_m$  of simplicial complexes such that  $B_1$  is the standard  $d$ -ball,  $B_m = C$  and, for  $2 \leq i \leq m$ ,  $B_i = B_{i-1} \cup \bar{\sigma}_i$  and  $B_{i-1} \cap \bar{\sigma}_i = \bar{\tau}_i$ , where  $\sigma_i$  is a  $d$ -simplex of  $B_i$  and  $\tau_i$  is a  $(d-1)$ -dimensional face of  $\sigma_i$ . A simplicial complex is called a *stacked  $(d-1)$ -sphere* if it is (combinatorially isomorphic to) the boundary of a stacked  $d$ -ball. An induction on  $m$  shows that a stacked  $d$ -ball triangulates the topological  $d$ -ball  $B^d$ , and hence a stacked  $(d-1)$ -sphere triangulates the  $(d-1)$ -sphere  $S^{d-1}$ . If  $C$  is a stacked ball then the dual graph  $\Lambda(C)$  is a tree. From [7, Lemma 2.1] we know the following result.

**Lemma 2.3** (Datta-Singh). *Let  $C$  be a pure simplicial complex of dimension  $d \geq 1$ .*

- (i) *If the dual graph  $\Lambda(C)$  is a tree then  $f_0(C) \leq f_d(C) + d$ .*
- (ii) *The dual graph  $\Lambda(C)$  is a tree and  $f_0(C) = f_d(C) + d$  if and only if  $C$  is a stacked  $d$ -ball.*

The following lemma follows from Lemmas 3.3 and 3.4 in [6] where it is proved in the more general setting of homology balls.

**Lemma 2.4.** *For  $d \geq 1$ , let  $\alpha$  be an interior  $(d-1)$ -face of a triangulated  $d$ -ball  $B$ . If  $\partial\bar{\alpha} \subseteq \partial B$  then the induced complex  $B[V \setminus \alpha]$  has exactly two connected components. Moreover, if  $\text{lk}_B(\alpha) = \{u, v\}$  then  $u$  and  $v$  are in different components of  $B[V(B) \setminus \alpha]$ .*

**Corollary 2.5.** *Let  $B$  be a stacked  $d$ -ball,  $d \geq 1$ . If  $\alpha$  is an interior  $(d-1)$ -face of  $B$  then the induced complex  $B[V \setminus \alpha]$  has exactly two connected components. Moreover, if  $\text{lk}_B(\alpha) = \{u, v\}$  then  $u$  and  $v$  are in different components of  $B[V(B) \setminus \alpha]$ .*

*Proof.* If  $B$  is a stacked  $d$ -ball then, by definition, the only interior faces in  $B$  are  $d$  and  $(d-1)$ -dimensional. Hence,  $\partial\bar{\alpha} \subseteq \partial B$ . The result now follows from Lemma 2.4.  $\square$

Finally, from [1, Lemma 4.6 and Remark 4.1] we know the following result.

**Lemma 2.6** (Bagchi-Datta). *Let  $S$  be a stacked  $(d-1)$ -sphere with edge graph  $G$  and  $d > 2$ . Let  $\overline{S}$  denote the simplicial complex whose faces are all the cliques of  $G$ . Then  $\overline{S}$  is a stacked  $d$ -ball and  $S = \partial\overline{S}$ . Moreover,  $\overline{S}$  is the unique stacked  $d$ -ball such that  $S = \partial\overline{S}$ .*

## 2.4 Bistellar moves

Bistellar moves are local combinatorial alterations of a triangulation which, in general, change the isomorphism type of the triangulation, but not the topology of the underlying space.

**Definition 2.7.** Let  $C$  be a pure  $d$ -dimensional (abstract) simplicial complex,  $\delta$  a  $(d-i)$ -face of  $C$ ,  $\gamma$  an  $i$ -face not contained in  $C$  such that  $\partial\gamma * \overline{\delta}$  is a subcomplex of  $C$ . Then the operation

$$C \mapsto (C \setminus \partial\gamma * \overline{\delta}) \cup \overline{\gamma} * \partial\delta$$

is called a *bistellar  $i$ -move* of  $C$ .

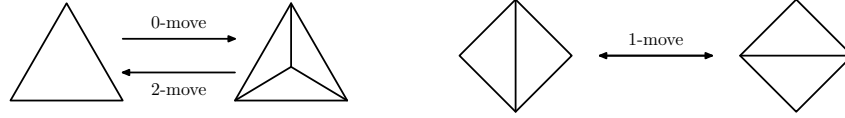


Figure 2.1: The bistellar moves in dimension two.

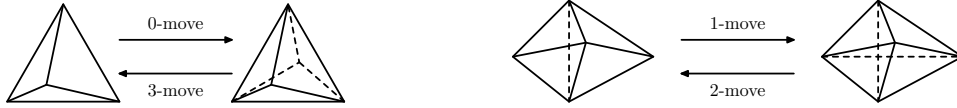


Figure 2.2: The bistellar moves in dimension three.

Note that, due to the specific set-up,  $\partial\gamma * \overline{\delta} \cup \overline{\gamma} * \partial\delta$  is isomorphic to the boundary complex of the  $(d+1)$ -simplex. A bistellar move thus replaces a triangulated  $d$ -ball with another triangulated  $d$ -ball with equal boundary. See Figures 2.1 and 2.2 for all types of bistellar moves in dimensions 2 and 3.

Note that 0-moves are sometimes called *stellar subdivisions*. In dimension 2, a 1-move is often referred to as *edge flip*. Moreover, a bistellar  $i$ -move in a  $d$ -dimensional complex is also called  $(i+1) - (d+1-i)$ -move, accounting for the fact that a bistellar  $i$ -move in dimension  $d$  removes  $(i+1)$  facets and replaces them with  $(d+1-i)$  facets.

For two PL triangulations  $M_1$  and  $M_2$  we write  $M_1 \sim M_2$  if there exist a sequence of bistellar moves turning  $M_1$  into a PL triangulation combinatorially isomorphic to  $M_2$ . Bistellar moves are a very powerful tool due to the following theorem by Pachner.

**Theorem 2.8** (Pachner [13]). *Two PL triangulations of  $d$ -manifolds  $M_1$  and  $M_2$  are PL homeomorphic if and only if  $M_1 \sim M_2$ .*

Accordingly, bistellar moves are sometimes referred to as *Pachner moves*. Due to Theorem 2.8 the following definition is well-defined.

**Definition 2.9.** The *Pachner graph* of a piecewise linear manifold  $M$  is the graph whose vertices are PL triangulations of  $M$  up to combinatorial isomorphism, with edges between all pairs of PL triangulations of  $M$  that can be transformed into isomorphic copies of each other by a single bistellar move.

We denote the Pachner graph of all  $n$ -vertex 2-spheres by  $\mathcal{P}_n$ . Note that all edges in  $\mathcal{P}_n$  correspond to edge flips. Given a triangulated 2-sphere  $S$  with edge  $ab$  and  $\text{lk}_S(ab) = \{x, y\}$ , then the edge flip  $S \mapsto S \setminus \{abx, aby\} \cup \{axy, bxy\}$  is valid if and only if  $xy$  is not an edge of  $S$ . It is denoted by  $ab \mapsto xy$ .

The Pachner graph of all  $n$ -vertex flag 2-spheres is denoted by  $\mathcal{F}_n$ , the Pachner graph of all Hamiltonian 2-spheres by  $\mathcal{H}_n$ , and the Pachner graph of all  $n$ -vertex stacked 2-spheres by  $\mathcal{S}_n$ . Note that, naturally, all of these graphs are induced subgraphs in the Pachner graph of the 2-sphere. In particular, a priori it is not clear, whether or not any of them is connected. The following statement is due to work by Mori, Nakamoto and Ota.

**Theorem 2.10** ([12]).  *$\mathcal{H}_n$  is connected and of diameter  $\leq 4n - 20$ .*

In this article, we focus on structural properties of  $\mathcal{F}_n$  and  $\mathcal{S}_n$ .

### 3 The Pachner graph $\mathcal{F}_n$ of $n$ -vertex flag 2-spheres

In this section we prove that, for  $n \geq 8$ , the Pachner graph  $\mathcal{F}_n$  of all  $n$ -vertex flag 2-spheres contains exactly two components, one of them consisting of the double cone  $\Gamma_n$ , the other one containing all other  $n$ -vertex flag 2-spheres. Throughout this section we write  $T \sim T'$  for two  $n$ -vertex flag 2-spheres meaning that there exists a sequence of edge flips connecting  $T$  and  $T'$  preserving the flagness property at each step. We prove the following statement.

**Theorem 3.1.** *If  $T$  and  $T'$  are two  $n$ -vertex flag 2-spheres distinct from  $\Gamma_n$ , then  $T \sim T'$ .*

See Figures 3.1 to 3.3 for illustrations of the Pachner graph  $\mathcal{F}_n$  for  $n \in \{8, 9, 10\}$ .



Figure 3.1: The Pachner graph  $\mathcal{F}_8$  of 8-vertex flag 2-spheres.

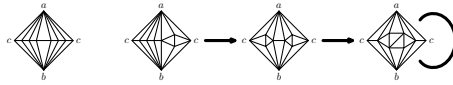


Figure 3.2: The Pachner graph  $\mathcal{F}_9$  arranged left to right by decreasing separation indices, see [5].

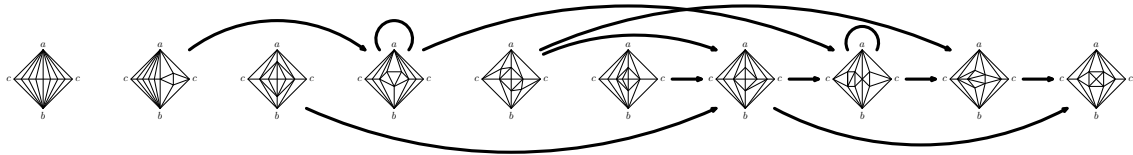


Figure 3.3: The Pachner graph  $\mathcal{F}_{10}$  arranged left to right by decreasing separation indices, see [5].

Before we can prove Theorem 3.1 we first need more terminology and a number of technical lemmas.

Let  $T$  be a flag 2-sphere. A subcomplex  $Q$  of  $T$  is called a *quadrilateral* if it is a triangulated disc and its boundary is a 4-cycle. A quadrilateral  $Q$  in  $T$  with boundary  $a-b-c-d-a$  is called *proper*, if  $a-b-c-d-a$  is an induced cycle in  $T$  and  $\deg_T(a), \deg_T(b), \deg_T(c), \deg_T(d) \geq 5$ . Since the boundary

is an induced cycle, a proper quadrilateral contains at least one interior vertex. A quadrilateral  $Q$  in  $T$  is called *ordered*, if it contains an interior vertex, and all of its interior vertices are of degree four. Since an ordered quadrilateral is a subcomplex of a flag 2-sphere, it follows that all the interior vertices lie on a path connecting diagonally opposite vertices of  $Q$ . We call this path a *diagonal path*, or just a *diagonal* of  $Q$ .

**Definition 3.2.** For  $n \geq 7$ , let  $A_n$  in  $\mathcal{F}_n$  be as in Figure 3.4 in the center. Note that  $A_7 = \Gamma_7$ , and  $A_n \neq \Gamma_n$  for  $n \geq 8$ .

For  $k \geq 3$ , let  $\mathcal{Q}_k$  be the triangulated quadrilateral with  $k$  interior vertices shown in Figure 3.4 on the right hand side. The path  $a_0 - a_1 - \dots - a_k$  is said to be the *diagonal path* of  $\mathcal{Q}_k$ .

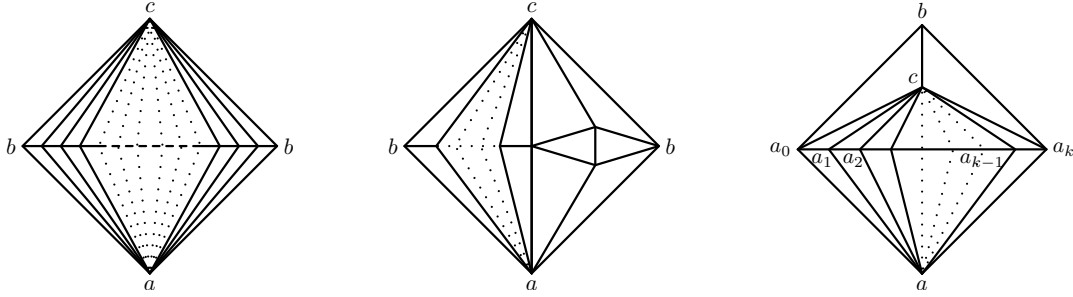


Figure 3.4: Left: the double cone  $\Gamma_n = S_{n-2}^1 * \{a, c\}$  over the  $(n-2)$ -gon. Center: the target  $n$ -vertex flag 2-sphere  $A_n$ . Right: quadrilateral  $\mathcal{Q}_k$  with boundary vertices  $a_0, a, a_k, b$  and interior vertices  $c, a_1, \dots, a_{k-1}$ .

**Lemma 3.3** (Transport Lemma). *Let  $T$  be a flag 2-sphere containing two ordered quadrilaterals  $\alpha$  and  $\beta$  with disjoint interiors, but a common boundary edge  $vw$ . Furthermore, let  $k \geq 2$  ( $\ell \geq 1$ ) be the number of interior vertices of  $\alpha$  (resp.,  $\beta$ ), and let  $v$  and  $w$  satisfy one of the following conditions:*

1.  $\deg_T(w) \geq 5$ , and the diagonal paths of  $\alpha$  and  $\beta$  intersect;
2.  $\deg_T(v) \geq 5$ ,  $\deg_T(w) \geq 6$ , the diagonal path of  $\alpha$  intersects  $v$ , and the diagonal path of  $\beta$  intersects  $w$ .

*Then there exists a flag 2-sphere  $T'$  such that (i)  $T \sim T'$ , (ii)  $T'$  contains two ordered quadrilaterals  $\alpha'$  and  $\beta'$ , (iii)  $T' = (T \setminus \{\alpha, \beta\}) \cup \{\alpha', \beta'\}$ , (iv)  $vw$  is a common edge of  $\alpha'$  and  $\beta'$  in  $T'$ , and (v) the number of interior vertices of  $\alpha'$  is  $k-1$ , and the number of interior vertices of  $\beta'$  is  $\ell+1$ .*

Lemma 3.3 gives precise conditions on when exactly we can “transport” an interior vertex of an ordered quadrilateral of  $T$  into an adjacent ordered quadrilateral without changing anything else in  $T$ . Note that all preconditions for Lemma 3.3 are fulfilled as soon as  $\alpha$  and  $\beta$  only share one edge.

*Proof.* Each ordered quadrilateral of  $T$  must be subdivided by a diagonal path containing all of its interior vertices all of which are of degree four. Hence, up to symmetry, there are two possible initial configurations to consider: The diagonal paths of  $\alpha$  and  $\beta$  either meet, or they run parallel with respect to  $vw$ .

**Case 1:** The diagonal paths of  $\alpha$  and  $\beta$  meet. W.l.o.g., let the diagonal paths meet at  $w$ . In this case, we have the sequence of edge flips shown in Figure 3.5 on the left hand side. Note that throughout this edge flip sequence the degrees of  $w$ ,  $v$ , and the other vertex of  $\alpha$  not on the diagonal path are, at some point, decreased to the initial degree minus one. The degrees of all other boundary vertices are never decreased below the initial degree. Since all three vertices of the

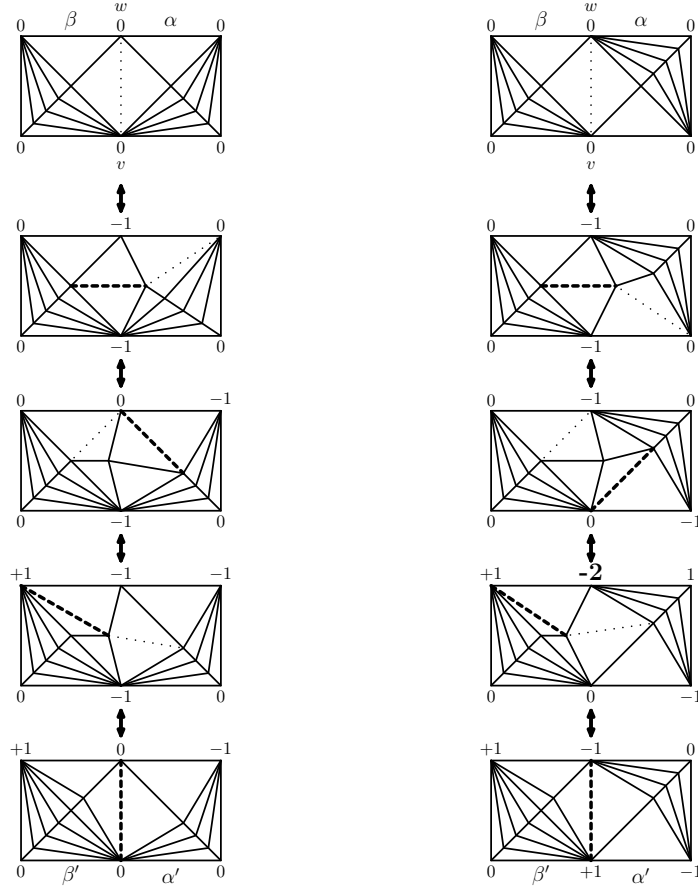


Figure 3.5: The Transport Lemma. Left: sequence of edge flips for intersecting diagonal paths. Right: sequence of edge flips for parallel diagonal paths. Both sequences “transport” an interior vertex of the quadrilateral  $\alpha$  into the adjacent quadrilateral  $\beta$ . No induced 3-cycles are introduced in the process.



former group are initially of degree  $\geq 5$  ( $w$  by assumption and the other two by the flagness of  $T$ ), the flagness condition is preserved in each step. Take a moment to verify that no other induced 3-cycle can be introduced by the flip sequence.

**Case 2:**  $\alpha$  and  $\beta$  have parallel diagonal paths. To comply with the labeling of the statement of the lemma, let the diagonal of  $\alpha$  intersect with  $v$  and the diagonal of  $\beta$  intersect with  $w$ . The sequence of edge flips is shown in Figure 3.5 on the right hand side. Note that, in this procedure, only the degree of  $w$  is, at one stage, decreased to the initial degree minus two. In addition,  $v$  and the other vertex of  $\alpha$  not on the diagonal path are, at some point, decreased to the initial degree minus one. The degrees of all other boundary vertices are never decreased below the initial degree. By assumption,  $w$  is of initial degree at least 6 and  $v$  is of initial degree 5. Again, the other vertex of  $\alpha$  not containing the diagonal must be of initial degree  $\geq 5$  by the flagness of  $T$ . It follows that the flagness condition is preserved in each step. Again, take a moment to verify that no other induced 3-cycle is introduced by the flip sequence.  $\square$

**Lemma 3.4.** *Let  $T$  be an  $n$ -vertex flag 2-sphere,  $n \geq 8$ , with induced 4-cycle  $a-a_0-b-a_k-a$  bounding  $\mathcal{Q}_k$ . Then either  $T = \Gamma_n$ ,  $T \sim A_n$ , or there exists an  $n$ -vertex flag 2-sphere  $T'$  with  $T \sim T'$ , such that (i)  $a-a_0-b-a_k-a$  is an induced 4-cycle in  $T'$  bounding an ordered quadrilateral  $Q$ , and (ii)  $T \setminus \mathcal{Q}_k = T' \setminus Q$ .*

*Proof.* Denote the vertices in the link of the lower vertex  $a$  by  $a_0-a_1-\dots-a_k$ , the apex above by  $c$ , and the upper vertex by  $b$ , see Figure 3.4 on the right hand side.

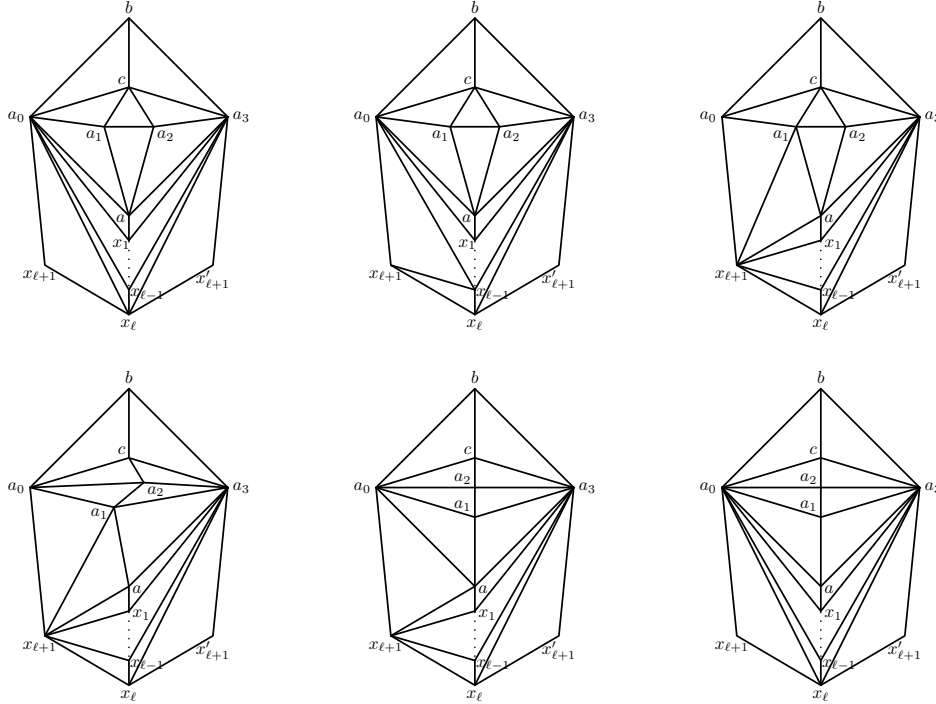


Figure 3.6: Resolving  $\mathcal{Q}_3$  into a quadrilateral with three interior vertices of degree 4.

**Case  $k = 3$ :** Add the outer two triangles  $a_0ax_1$  and  $a_3ax'_1$  to  $\mathcal{Q}_3$ . If  $x_1 = x'_1$  (i.e.,  $a$  is of degree 5) continue adding outer triangles  $a_0x_ix_{i+1}$  and  $a_3x_ix'_{i+1}$  until we reach vertices  $x_{\ell+1} \neq x'_{\ell+1}$  (i.e.,  $x_\ell$  is of degree  $\geq 5$ ), or  $x'_\ell = x_\ell = b$ .

If  $x'_\ell = x_\ell = b$ ,  $\ell \geq 2$ ,  $T$  must be isomorphic to  $A_{\ell+6}$  and we are done. Otherwise, there are two triangles  $a_0x_\ell x_{\ell+1}$  and  $a_3x_\ell x'_{\ell+1}$ ,  $x'_{\ell+1} \neq x_{\ell+1}$ . Moreover, neither  $a_0x'_{\ell+1}$  nor  $a_3x_{\ell+1}$  can be edges since otherwise  $T$  contains an induced 3-cycle. Hence, we can perform edge flips  $a_0x_\ell \mapsto x_{\ell+1}x_{\ell-1}$ ,  $a_0x_{\ell-1} \mapsto x_{\ell+1}x_{\ell-2}$ , etc. all the way down to  $a_0a \mapsto x_{\ell+1}a_1$ .

It follows that we can perform flips  $a_1c \mapsto a_0a_2$  and  $aa_2 \mapsto a_1a_3$ , followed by the initial sequence of edge flips in reverse, i.e.,  $x_{\ell+1}a_1 \mapsto a_0a$ ,  $x_{\ell+1}a \mapsto a_0x_1$ ,  $x_{\ell+1}x_1 \mapsto a_0x_2$ , all the way up to  $x_{\ell+1}x_{\ell-1} \mapsto a_0x_\ell$ . Observe that now all vertices inside  $\mathcal{Q}_3$  are of degree 4. This proves the result for  $k = 3$ . See Figure 3.6 for details of this flip sequence.

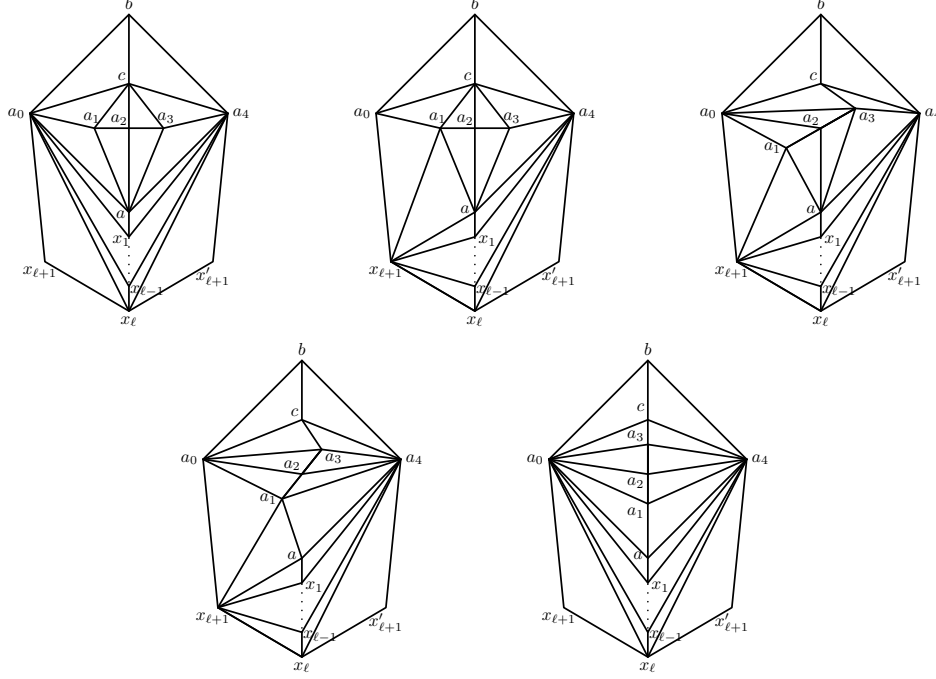


Figure 3.7: Resolving  $\mathcal{Q}_4$  into a quadrilateral with three interior vertices of degree 4.

**Case  $k = 4$ :** The case  $k = 4$  is very similar to the case  $k = 3$ . If  $x'_1 = x_1 = b$ , then  $T = \Gamma_8$ . If  $x'_\ell = x_\ell = b$ ,  $\ell \geq 2$ ,  $T$  decomposes into two ordered proper quadrilaterals along induced 4-cycle  $a_0-a-a_4-c-a_0$  for which we can use Lemma 3.3 to transport  $a_1$  or  $a_3$  away from its quadrilateral to conclude that  $T \sim A_n$ ,  $n \geq 8$ . Otherwise, the initial and the final sequence are identical. Once flip  $a_0a \mapsto x_{\ell+1}a_1$  is performed, we can perform  $a_1c \mapsto a_0a_2$  and  $a_2c \mapsto a_0a_3$ , followed by  $a_3a \mapsto a_2a_4$  and  $a_2a \mapsto a_1a_4$ . Reversing the initial sequence then yields the desired result, see Figure 3.7 for details.

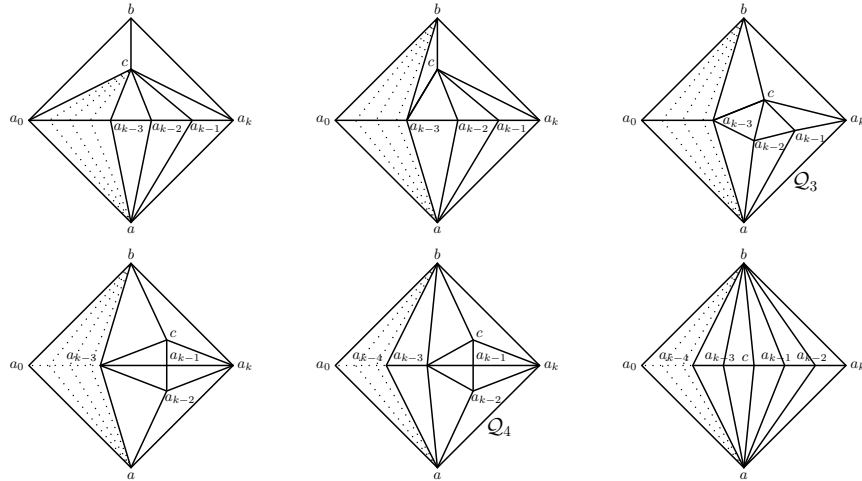


Figure 3.8: Resolving  $\mathcal{Q}_k$ ,  $k > 4$ , into a quadrilateral with  $k$  interior vertices of degree 4.

**Case  $k > 4$ :** From  $k > 4$  it follows that  $n \geq 10$ . Moreover,  $a_0 \neq a_k$ , and  $a_0a_k$  is a non-edge of  $T$  since  $a_0-a-a_k-b-a_0$  is an induced 4-cycle. We start by performing flips  $a_0c \mapsto ba_1$ ,  $a_1c \mapsto ba_2$ , all the way to  $a_{k-4}c \mapsto ba_{k-3}$ . The quadrilateral then splits into two parts. One with only degree 4 interior vertices (at least one), the other one being isomorphic to  $\mathcal{Q}_3$  with diagonal path going from  $a_{k-3}$  to  $a_k$ . Use the case  $k = 3$  to turn  $\mathcal{Q}_3$  into a quadrilateral containing only interior vertices of degree 4 with the diagonal path running from  $a$  to  $b$ . Since  $k > 4$ , the overall quadrilateral again splits into two parts, one with only degree 4 interior vertices (possibly none), the other one being isomorphic to  $\mathcal{Q}_4$  with diagonal path going from  $a$  to  $b$ . Use the case  $k = 4$  to either conclude that  $T \sim A_n$ , or to turn  $\mathcal{Q}_4$  into a quadrilateral containing only degree 4 interior vertices and diagonal running from  $a_{k-4}$  to  $a_k$ . In the latter case the overall quadrilateral now only has interior vertices of degree 4 which proves the lemma. See Figure 3.8 for details.  $\square$

**Lemma 3.5** (Merge Lemma). *Let  $T$  be an  $n$ -vertex flag 2-sphere containing two ordered quadrilaterals  $\alpha$  and  $\beta$  with disjoint interiors, but common outer edges  $uv$  and  $w$ . Then either  $T = \Gamma_n$ ,  $T \sim A_n$ , or  $T \sim T'$  where  $T'$  has an ordered quadrilateral  $\gamma$  with boundary  $\partial(\alpha \cup \beta)$  and  $T' = (T \setminus \{\alpha, \beta\}) \cup \{\gamma\}$ .*

*Proof.* We have four cases for the initial configuration of  $\alpha$  and  $\beta$ , shown in Figure 3.9.

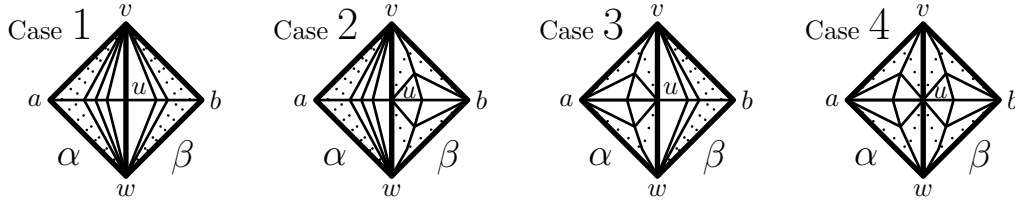


Figure 3.9: The four initial configurations for  $\alpha$  and  $\beta$  in the proof of Lemma 3.5.

**Case 1:** If  $a = b$  then  $\alpha \cup \beta = T$ . In this case,  $T = \Gamma_n$  with cone apices  $v$  and  $w$  and we are done.

If  $a \neq b$  we can merge  $\alpha$  and  $\beta$  into one larger ordered quadrilateral with boundary  $\partial(\alpha \cup \beta)$ .

**Case 2:** Again, if  $a = b$  then  $\alpha \cup \beta = T$ . If, in this case,  $\beta$  only contains one interior vertex then we have  $\Gamma_n$  with cone apices  $v$  and  $w$  and we are done. If  $\alpha$  only contains one interior vertex, both  $v$  and  $w$  are of degree 4, and we have  $\Gamma_n$  with cone apices  $u$  and  $a = b$ . Thus, we can assume both  $\alpha$  and  $\beta$  have at least two interior vertices. In this case, we iteratively apply Lemma 3.3 to transport interior vertices from  $\beta$  to  $\alpha$  across edge  $uw$  until  $u$  is of degree 5 and we obtain  $A_n$ .

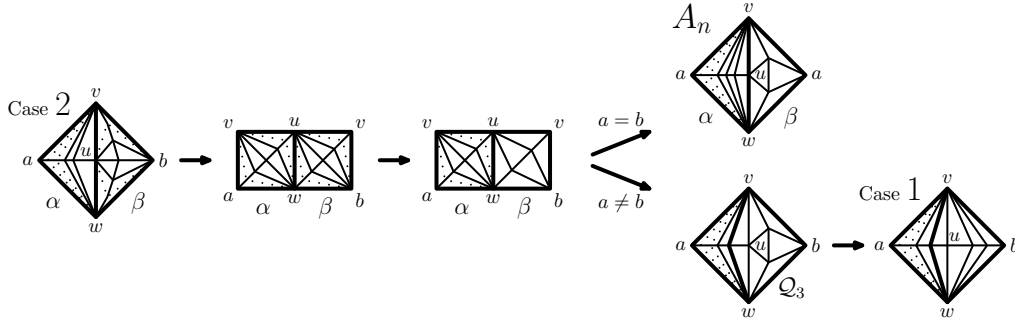


Figure 3.10: Transporting vertices in Case 2.

If  $a \neq b$ , we, again, apply Lemma 3.3 to transport interior vertices from  $\beta$  to  $\alpha$  across edge  $uw$  until  $u$  is of degree 5. The quadrilateral  $\beta$  together with the two rightmost triangles of  $\alpha$  now form a quadrilateral isomorphic to  $\mathcal{Q}_3$  with diagonal path from  $v$  to  $w$  (see Figure 3.10). This can be resolved into a quadrilateral with interior vertices all of degree 4 and diagonal intersecting  $b$  (note

that  $a$  is of degree  $> 4$  and thus (i) the preconditions of Lemma 3.4 are satisfied and (ii) we can always resolve  $\mathcal{Q}_3$  in this case) and we are back to Case 1.

**Case 3:** This is completely analogue to Case 2.

**Case 4:** Again, if  $a = b$  then  $\alpha \cup \beta = T$ , and we have  $T = \Gamma_n$  with cone apices  $u$  and  $a = b$ .

Hence, let  $a \neq b$ . If  $\alpha$  only contains a single interior vertex we fall back to Case 2, if  $\beta$  only contains a single vertex we fall back to Case 3. Thus we can assume both  $\alpha$  and  $\beta$  have at least two interior vertices. In this case,  $\deg_T(u) \geq 6$ ,  $\deg_T(v), \deg_T(w) \geq 5$ , and we apply Lemma 3.3 to transport vertices from  $\alpha$  to  $\beta$  until  $\alpha$  only contains a single interior vertex. Then we proceed with Case 2.  $\square$

**Lemma 3.6.** *For  $n \geq 8$ , let  $T \in \mathcal{F}_n \setminus \{\Gamma_n\}$ . Then there exists  $T' \in \mathcal{F}_n \setminus \{\Gamma_n\}$  with  $T \sim T'$ , and  $a, b, c, d \in V(T')$  such that (i)  $a-b-c-d-a$  is an induced 4-cycle, and (ii)  $\deg_{T'}(a), \deg_{T'}(b), \deg_{T'}(c), \deg_{T'}(d) \geq 5$ . In particular,  $T'$  splits into two proper quadrilaterals  $Q \cup_{\partial} R = T'$ .*

*Proof.* If  $T$  contains a vertex  $v$  of degree 4, then, by the flagness of  $T$ , the link of  $v$  is an induced 4-cycle, say  $a-b-c-d-a$ . If any of these vertices, say  $a$ , is of degree 4, then, since  $n \geq 8$ , the boundary of the union of the stars  $v$  and  $a$  is an induced 4-cycle. Moreover,  $b$  and  $d$  are of degree  $\geq 5$ . Iterating this process either yields an induced 4-cycle  $x-b-c-d-x$ , for some vertex  $x$  of  $T$  of degree  $\geq 5$ , or  $x = d$ , and  $T$  is isomorphic to  $\Gamma_n$ , a contradiction. Repeating the above procedure for  $c$  yields an induced 4-cycle  $x-b-y-d-x$  in  $T$  (note that  $x \neq y$  since the degree of  $x$  is at least 5).

If  $T$  does not contain a vertex of degree 4, then  $T$  must contain a vertex  $w$  of degree 5. Let  $auw$  and  $buw$  be two adjacent triangles in the star of  $w$ . If  $a$  and  $b$  have a common neighbour  $x$  distinct from  $w$  and  $u$ , then  $x-a-w-b-x$  is an induced 4-cycle, and we are done since  $T$  has no vertex of degree 4. Otherwise the flip  $uw \mapsto ab$  yields a flag 2-sphere in which  $w$  has degree 4. Now the link of  $w$  is an induced 4-cycle with all four vertices being of degree  $\geq 5$ .  $\square$

**Lemma 3.7.** *Let  $T$  be an  $n$ -vertex flag 2-sphere which splits into two proper quadrilaterals  $Q \cup_{\partial} R = T$  along an induced 4-cycle  $\partial = a-c-b-d-a$ . Furthermore, let  $k > 0$  be the number of interior vertices of  $Q$ . Then there exists an  $n$ -vertex flag 2-sphere  $T'$  with  $T \sim T'$ , such that  $T' = Q' \cup_{\partial} R$ , and the interior of  $Q'$  only contains degree four vertices.*

Note that, in  $T'$ , neither  $Q'$  nor  $R$  need to be proper quadrilaterals. However, both  $Q'$  and  $R$  contain interior vertices. In particular, each of  $a, b, c$ , and  $d$  is contained in at least two triangles of both  $Q'$  and  $R$ . We deal with this issue separately whenever we need to, namely in the proof of Theorem 3.1.

*Proof.* We prove this statement by induction on the number  $k$  of interior vertices in  $Q$ . First note that  $k > 0$ , and that the statement is true for  $k \leq 2$ .

Let  $a-c-b-d-a$  be the boundary of a quadrilateral  $Q$  in  $T$  with  $k \geq 3$  interior vertices, such that  $\deg_T(a), \deg_T(b), \deg_T(c), \deg_T(d) \geq 5$ . Since  $a-c-b-d-a$  is induced,  $ab$  and  $cd$  cannot be edges of  $T$ .

**Claim:** There exist a triangulation  $T'$  with  $T \sim T'$ , such that  $T' = Q' \cup_{\partial} R$ , and in the interior of  $Q'$  either  $a$  and  $b$  or  $c$  and  $d$  have at least one common neighbour.

**Proof of the claim:** In the following procedure we always refer to the flag 2-sphere as  $T$  and to the quadrilateral enclosed by  $a-a_0-b-a_m-a$  as  $Q$ , although both objects are altered in the process.

1. We denote all neighbours of  $a$  in  $Q$  from left to right by  $c = a_0, a_1, \dots, a_m = d$ .
2. If  $a_0$  and  $a_m$  have a common neighbour in  $Q$  other than  $a$  and  $b$  we are done.
3. If no such neighbour exists, let  $1 \leq j \leq m-1$  be the largest index for which  $a_0$  and  $a_j$  have common neighbours outside the star of  $a$ .

By the planarity of  $Q$ , there exist an outermost neighbour  $x$  in  $Q$ , bounding a quadrilateral  $x-a_0-a-a_j-x$  that contains all other common neighbours of  $a_0$  and  $a_j$ . Note that, in this case,  $a_j$  must be of degree  $\geq 5$ . If  $x = b$ ,  $a$  and  $b$  have a common neighbour and we are done. Otherwise,  $a_0$  must be of degree  $\geq 5$ .

4. If  $x-a_0-a-a_j-x$  does not contain interior vertices, we must have  $j = 1$  and the quadrilateral consist of the two triangles  $a_0a_1a$  and  $a_0a_1x$ . Note that  $x \neq a_i$  by the flagness of the triangulation, and  $xa_i$  is a non-edge for  $2 \leq i \leq m$  by construction of the procedure.

Now  $a_0$  and  $a_1$  both must be of degree  $\geq 5$ , and  $a$  and  $x$  do not have common neighbours other than  $a_0$  and  $a_1$ . Hence, we can perform flip  $a_0a_1 \mapsto ax$ , and start over at step 1 with  $a'_0 = a_0, a'_1 = x, a'_2 = a_1, \dots, a'_{m+1} = a_m$ .

5. If the quadrilateral bounded by  $x-a_0-a-a_j-x$  contains interior vertices, we have  $\deg_T(x) \geq 5$  and, in particular, it is a proper quadrilateral with less interior vertices than  $Q$ . We can thus use the induction hypothesis to rearrange the interior of  $x-a_0-a-a_j-x$  to only contain interior vertices of degree 4. Note that all of its four boundary vertices still must have degree at least 5 (i.e., the rearranged quadrilateral is an ordered proper quadrilateral). This is important later on in the proof.
6. After rearranging the proper quadrilateral  $x-a_0-a-a_j-x$ ,  $j > 1$ , into an ordered proper quadrilateral, repeat steps 3 to 5 by looking for the largest index  $j < \ell \leq m-1$  for which  $a_j$  and  $a_\ell$  have common neighbours outside the star of  $a$ . Note that, whenever we flip an edge in step 4 we start over at step 1 with a strictly larger degree of vertex  $a$  in  $Q$ .

This process either yields the desired result, or it terminates with  $Q$  having a sequence of smaller ordered quadrilaterals  $Q_1, \dots, Q_p$  around vertex  $a$ ,  $p > 1$ .

Call the “peaks” of the quadrilaterals  $x_1, \dots, x_p$ , and the “valleys” between quadrilaterals  $a_0 = y_0, \dots, y_p = a_m$  (see Figure 3.11 for a picture of this situation). By construction, all  $x_i$ ,  $1 \leq i \leq p$ , and  $y_j$ ,  $0 \leq j \leq p$  are of degree  $\geq 5$  (see step 5 above). That is, the quadrilaterals  $Q_i$ ,  $1 \leq i \leq p$ , are ordered and proper.

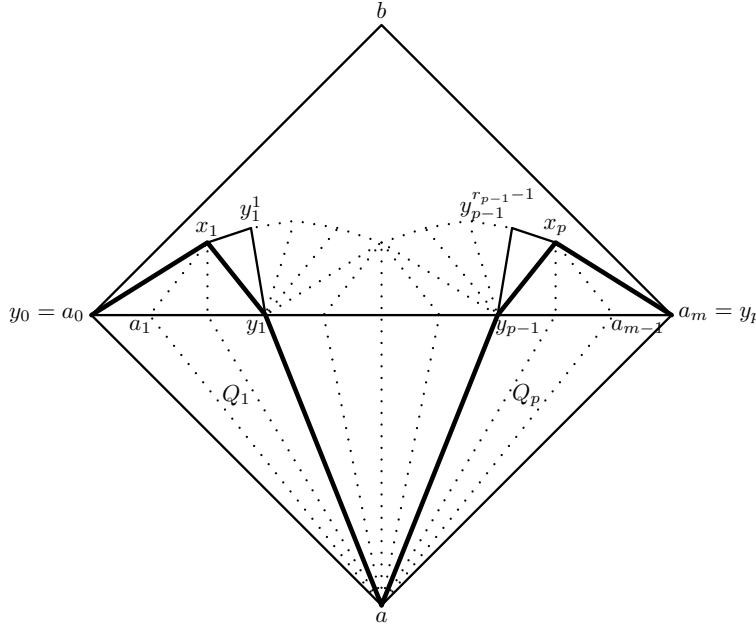


Figure 3.11: The quadrilateral  $Q$  after performing steps 1-6, and after reorganising the interior vertices of quadrilaterals  $Q_i$ ,  $1 \leq i \leq p$ .

Recall that all quadrilaterals  $Q_i$ ,  $1 \leq i \leq p$ , only contain degree 4 interior vertices. We want all of the diagonal paths of  $Q_i$ ,  $1 \leq i \leq p$ , to run from  $y_{i-1}$  to  $y_i$ . If  $Q_i$  only has one interior vertex, this is always the case. Thus, assume that there exist a pair of quadrilaterals  $Q_i$  and  $Q_{i+1}$ ,  $1 \leq i \leq p-1$ , sharing common edge  $ay_i$ , and, w.l.o.g., assume that  $Q_i$  has a diagonal path from  $a$  to  $x_i$  of length  $\geq 2$ .

Observe that in this particular situation, both  $a$  and  $y_i$  must be of degree  $\geq 6$ . Hence we can apply Lemma 3.3 to “transport” all but one interior vertices of  $Q_i$  to the diagonal path of  $Q_{i+1}$ , and declare the diagonal path in  $Q_i$  to run from  $y_{i-1}$  to  $y_i$ . If the diagonal path of  $Q_{i+1}$  connects  $y_i$  with  $y_{i+1}$  we are done. If not, note that, again, both  $a$  and  $y_i$  must be of degree  $\geq 6$ . We proceed by transporting all interior vertices of  $Q_{i+1}$  but one onto the new diagonal path from  $y_{i-1}$  to  $y_i$  of  $Q_i$ , and declare the diagonal path in  $Q_{i+1}$  to run from  $y_i$  to  $y_{i+1}$ . Repeating this with all pairs of quadrilaterals containing at least one diagonal intersecting  $a$  yields the desired result.

Denote the vertices in the upper link of  $y_j$  by  $x_j = y_j^0, y_j^1, y_j^2, \dots, y_j^{r_j} = x_{j+1}$ , see Figure 3.11. By construction we have  $r_j > 0$ .

Since  $p > 1$ ,  $x_1, y_1 = a_j$ , and  $x_2$  are in the interior of  $Q$ . Moreover, both  $a_j = y_1$ ,  $j > 1$ , and  $x_2$  are of degree  $\geq 5$  and, by design of the procedure,  $y_0 y_1^\ell$ ,  $1 \leq \ell \leq r_1$ , is a non-edge (otherwise  $y_1^\ell$  would have been a better choice for  $x_1$ ). It follows that we can perform the flips  $x_1 a_j \mapsto a_{j-1} y_1^\ell$ ,  $x_1 a_{j-1} \mapsto a_{j-2} y_1^\ell$ , etc., all the way down to  $x_1 a_2 \mapsto a_1 y_1^\ell$ . Note that  $y_1$  and  $x_1$  are now both of degree  $\geq 4$ , the degree of  $y_1^\ell$  is larger than before,  $a_1$  is of degree 5, and all other degrees have not changed. Since  $x_1 a_i$ ,  $2 \leq i \leq m$ , must be non-edges,  $a$  and  $x_1$  do not have common neighbours. We can thus perform the flip  $a_0 a_1 \mapsto a x_1$ , see Figure 3.12 for details.

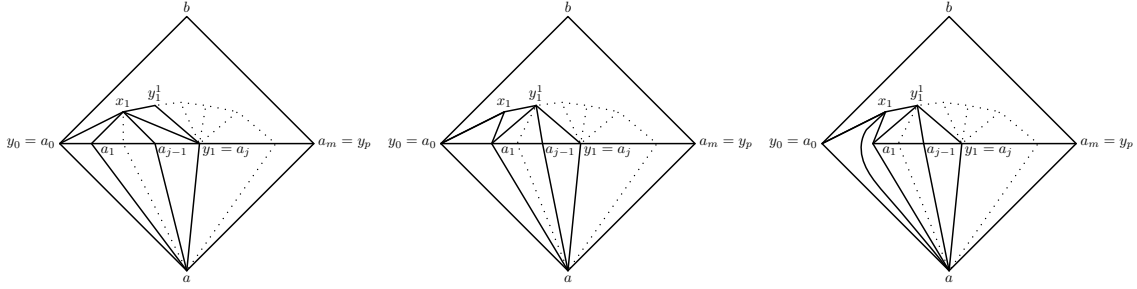


Figure 3.12: Increasing the size of the link of  $a$ .

This increases the degree of  $a$ . We now start over with our procedure at step 1.

Since there are only finitely many vertices inside  $Q$ , this procedure must terminate with  $Q$  containing a common neighbour of  $a$  and  $b$  or  $a_0$  and  $a_m$  as interior vertex. This proves the claim.

We can thus assume that we have an  $n$ -vertex flag 2-sphere  $T'$ ,  $T \sim T'$ , such that either  $a$  and  $b$  or  $c$  and  $d$  have at least one common neighbour.

Assume that there exist at least one common neighbour of  $a$  and  $b$  (the case that  $c$  and  $d$  have at least one common neighbour is completely analogue). If all such neighbours are of degree 4, all interior vertices must be neighbours of  $a$  and  $b$ , and of degree 4. In particular, we are done. Otherwise, choose a common neighbour  $e$  of degree  $\geq 5$ , and split  $Q$  into two smaller quadrilaterals  $Q_1$  and  $Q_2$  with boundaries  $e-a-a_0-b-e$  and  $e-a-a_m-b-e$ . If both  $Q_1$  and  $Q_2$  have interior vertices, use the induction hypothesis to transform  $Q_i$  to  $Q'_i$ ,  $i = 1, 2$ , in which all interior vertices have degree 4. Note that vertices  $a$  and  $b$  will always be of degree  $\geq 6$  in this setting, even after applying the induction hypothesis. It follows that the induction hypothesis can be applied to both  $Q_1$  and  $Q_2$  without violating any preconditions after the first application. Use Lemma 3.5 to merge both quadrilaterals, or conclude that  $T \sim A_n$ . We have that  $T \not\sim \Gamma_n$ , since  $\Gamma_n$  does not split into two proper quadrilaterals, as required by the statement of Lemma 3.7.

Hence, w.l.o.g., let  $Q_1$  be without interior vertices. Use the induction hypothesis to transform  $Q_2$  into  $Q'_2$  with only degree 4 vertices inside. Now either  $a_j$  is of degree four, all interior vertices of  $Q' = Q_1 \cup Q'_2$  are of degree 4, and we are done. Or  $Q'$  is isomorphic to  $Q_k$ , and can be transformed into a quadrilateral containing only degree 4 vertices by Lemma 3.4 (or  $T \sim A_n$ ).  $\square$

*Proof of Theorem 3.1.* To prove the theorem it suffices to show that  $T \sim A_n$  for all  $T \in \mathcal{F}_n \setminus \{\Gamma_n\}$ .

Apply Lemma 3.6 to split  $T$  into two proper quadrilaterals  $T = Q \cup_\partial R$ . This is always possible since  $T \neq \Gamma_n$ . Use Lemma 3.7 to turn all interior vertices of both  $Q$  and  $R$  into vertices of degree 4.

Should, after the first or second application of Lemma 3.7, any of the boundary vertices of  $Q$  (or  $R$ ) be of degree four, we can grow  $Q$  (or  $R$ ) such that eventually it is bounded by vertices of degree  $\geq 5$ , or  $T \sim \Gamma_n$ . However, since all edge flips on  $\Gamma_n$  produce a non-flag 2-sphere triangulation, the latter case implies  $T = \Gamma_n$ , a contradiction.

Thus,  $T$  can be transformed into a triangulation  $T''$  of the 2-sphere which splits into two ordered proper quadrilaterals. This corresponds to the cases  $a = b$  in the proof of Lemma 3.3. In particular, either  $T'' = \Gamma_n$ , which is impossible,  $T'' = A_n$ , or the degrees of all vertices of the separating induced 4-cycle satisfy the preconditions of Lemma 3.3, and we can conclude that  $T \sim A_n$ .  $\square$

## 4 The Pachner graph $\mathcal{S}_n$ of $n$ -vertex stacked 2-spheres

Every pair of  $n$ -vertex stacked 2-spheres is, by definition, connected in the Pachner graph of all stacked 2-spheres by a sequence of  $(n - 4)$  2-moves, followed by a sequence of  $(n - 4)$  0-moves. However, if we look at the Pachner graph  $\mathcal{S}_n$  of  $n$ -vertex stacked 2-spheres, the situation is much different.

In this section, we show that the structure of  $\mathcal{S}_n$  is very special. More precisely, we prove that  $\mathcal{S}_n$  is not connected for  $n \geq 7$ , see Corollary 4.6, and that the number of connected components rapidly increases with the number of vertices, see Corollary 4.7. See Table 1 for the number and sizes of connected components of  $\mathcal{S}_n$  for  $n \leq 14$ .

$n$	$\#(\mathcal{S}_n)$	$\#$ cc	size of connected components
4	1	1	1
5	1	1	1
6	1	1	1
7	3	1	3
8	7	2	1, 6
9	24	2	1, 23
10	93	3	3, 4, 86
11	434	5	1, 7, 10, 19, 397
12	2110	8	1, 2, 6, 43, 46, 57, 82, 1873
13	11002	15	1, 2, 2, 3, 4, 6, 6, 7, 57, 222 223, 246, 326, 394, 9503
14	58713	33	1, 1, 3, 4, 4, 4, 5, 6, 6, 6, 6, 7, 7, 9, 9, 12, 15, 19, 27, 28, 36, 36, 246, 304, 339, 757, 1165, 1182, 1571, 1944, 1987, 48958

Table 1: Sizes of the connected components of  $\mathcal{S}_n$  for  $n \leq 14$ .

For a stacked 2-sphere, let  $\bar{S}$  be the unique stacked 3-ball whose boundary is  $S$ , see Lemma 2.6. If  $\alpha$  is a triangle of  $S$  then  $\alpha$  is a face of a unique tetrahedron of  $\bar{S}$  (clique in the edge graph of  $S$ ). We denote this unique tetrahedron by  $\tilde{\alpha}$ . Naturally,  $\tilde{\alpha}$  is a node in the dual graph  $\Lambda(\bar{S})$ .

**Theorem 4.1.** *Let  $S$  be a stacked 2-sphere. Let  $\alpha = abc$ ,  $\beta = abd$  be two triangles of  $S$ . Let  $\tilde{\alpha}$  (resp.,  $\tilde{\beta}$ ) be the unique tetrahedron in  $\bar{S}$  containing  $\alpha$  (resp.,  $\beta$ ). Then both  $cd$  is not an edge of  $S$  and the 2-sphere  $T$  obtained from  $S$  by the edge flip  $ab \mapsto cd$  is stacked if and only if the nodes  $\tilde{\alpha}$  and  $\tilde{\beta}$  of  $\Lambda(\bar{S})$  are adjacent in  $\Lambda(\bar{S})$ .*

*Remark 4.2.* For an edge flip  $ab \mapsto cd$  on a stacked 2-sphere  $S$ , we must have  $abc, abd \in S$  and  $cd \notin S$ . Then the link of  $ab$  in  $\bar{S}$  is a path from  $c$  to  $d$  of length  $\geq 2$ . Lemma 2.6 and the proof of Theorem 4.1 imply that an  $n$ -vertex 2-sphere  $T$  obtained from a stacked 2-sphere  $S$  by an edge flip  $ab \mapsto cd$  is stacked if and only if one of the following conditions holds:

- The path from  $c$  to  $d$  is of length exactly two.
- Edge  $ab$  is contained in exactly two tetrahedra of  $\bar{S}$ .
- The vertices  $a$  and  $b$  have exactly three common neighbours in  $S$ .
- There exists a unique vertex  $e \notin \{c, d\}$  such that  $ae$  and  $be$  are edges of  $S$ .

While some of these conditions are easier to grasp, others are more efficient for implementations. It is thus useful to keep all of them in mind.

*Proof.* Suppose  $\tilde{\alpha}$  and  $\tilde{\beta}$  are adjacent in the dual graph  $\Lambda(\bar{S})$ ,  $\tilde{\alpha} \neq \tilde{\beta}$ . Then there exists a vertex  $e$  of  $S$  such that  $\tilde{\alpha} = abce$  and  $\tilde{\beta} = abde$  ( $e \notin \{d, c\}$  since  $\tilde{\alpha} \neq \tilde{\beta}$ ). If  $cd$  is an edge of  $S$  then  $\{a, b, c, d, e\}$  is a clique in the edge graph of  $S$  and hence, by Lemma 2.6,  $abcde$  is a simplex of  $\bar{S}$ . This is not possible since  $\bar{S}$  is 3-dimensional.

Let  $B = \bar{S} \cup abcd$ . Since  $\bar{S} \cap abcd$  is a 2-disk,  $B$  is a triangulated 3-ball. The link  $\text{lk}_B(ab)$  is the induced 3-cycle  $c-d-e-c$  in  $B$ . Let  $D$  be obtained from  $B$  by the bistellar 2-move that replaces the three tetrahedra around edge  $ab$  with two tetrahedra sharing triangle  $cde$ , or  $ab \mapsto cde$  in short. Then  $D$  is a 3-ball with tetrahedra  $\tilde{\gamma} = acde, \tilde{\delta} = bcde \in D$ . Note that  $\partial D = T$ , where  $T$  is the 2-sphere obtained from  $S$  by the edge flip  $ab \mapsto cd$  (and all edges of  $D$  are boundary edges).

If  $\xi$  is a tetrahedron of  $\bar{S}$  adjacent to the node  $\tilde{\alpha}$  in  $\Lambda(\bar{S})$  then  $\xi$  is of the form  $acex$  or  $bcey$  for some vertex  $x$  (note that  $abc \in \partial B$ ). In the first case,  $\xi$  is adjacent to the node  $\tilde{\gamma}$  in  $\Lambda(\bar{T})$  and in the second case,  $\xi$  is adjacent to the node  $\tilde{\delta}$  in  $\Lambda(\bar{T})$  (see Figure 4.1 and Remark 4.3 below). Similarly, a neighbour of  $\tilde{\beta}$  in  $\Lambda(\bar{S})$  is adjacent to exactly one of  $\tilde{\gamma}$  and  $\tilde{\delta}$  in  $\Lambda(\bar{T})$ . Since  $\tilde{\gamma}$  is adjacent to  $\tilde{\delta}$  in  $\Lambda(\bar{T})$ , it follows that  $\Lambda(D)$  is connected and has exactly as many arcs and nodes as  $\Lambda(\bar{S})$ . Hence  $\Lambda(D)$  is a tree.

Now,  $f_0(D) = f_0(\bar{S}) = f_3(\bar{S}) + 3 = f_3(D) + 3$ . Therefore, by Proposition 2.3 (ii),  $D$  is a stacked 3-ball and hence  $T = \partial D$  is a stacked 2-sphere.

Conversely, suppose  $cd$  is not an edge of  $S$  and the triangulated 2-sphere  $T$  obtained from the stacked 2-sphere  $S$  by the edge flip  $ab \mapsto cd$  is a stacked 2-sphere. Let the vertex set of  $S$  (and  $T$ ) be  $V$ . Observe that both  $\gamma = acd$  and  $\delta = bcd$  are triangles of  $T$ .

Since  $ab, abc, abd \in S = \partial \bar{S}$ ,  $\text{lk}_{\bar{S}}(ab)$  is a path in  $\text{Skel}_1(\bar{S})$  from  $c$  to  $d$ . Let  $\text{lk}_{\bar{S}}(ab) = e_0 - e_1 - \dots - e_k - e_{k+1}$  for some  $k \geq 1$ , where  $e_0 = c$  and  $e_{k+1} = d$ . We have that  $abce_1 = abe_0e_1$ ,  $abe_1e_2, \dots, abe_{k-1}e_k$ ,  $abe_ke_{k+1} = abde_k$  are tetrahedra in  $\bar{S}$ . Thus,  $abe_1, \dots, abe_k$  are interior triangles of  $\bar{S}$ . By Corollary 2.5,  $B[V \setminus \{a, b, e_1\}]$  has two components, one contains  $e_0$  and the other contains  $e_2$ . Thus, the common neighbours of  $e_0$  and  $e_2$  – in  $\text{Skel}_1(\bar{S}) = \text{Skel}_1(S)$  – are  $a, b$  and  $e_1$ . Similarly, the set of common neighbours of  $e_{i-1}$  and  $e_{i+1}$  is  $\{a, b, e_i\}$  for  $1 \leq i \leq k$ . This implies that the set of common neighbours of  $c = e_0$  and  $d = e_{k+1}$  in  $\text{Skel}_1(T)$  is  $\{a, b, e_1\} \cap \{a, b, e_k\}$ .

The triangles  $\gamma = acd = ae_0e_{k+1}$  and  $\delta = bcd = be_0e_{k+1}$  are contained in unique tetrahedra  $\tilde{\gamma} = acdx$  and  $\tilde{\delta} = bcdy$  of  $\bar{T}$  which are cliques in  $\text{Skel}_1(T)$ . Hence,  $a, b, x$  and  $y$  are common neighbours of  $c$  and  $d$ , which, by the above, is only possible if  $e := x = y = e_1 = e_k$  (note that  $ab$  is a non-edge in  $T$ , and thus  $\{a, b, c, d, e\}$  is not a clique in  $\text{Skel}_1(T)$ ). This implies that  $\{a, b, c, e\}$  and  $\{a, b, d, e\}$  are cliques in  $\text{Skel}_1(S)$ . It follows that, by Lemma 2.6,  $abce$  and  $abde$  are tetrahedra in  $\bar{S}$ . Since  $\alpha = abc$  (resp.  $\beta = abd$ ) is contained in a unique tetrahedron of  $\bar{S}$ , it follows that  $\tilde{\alpha} = abce$  ( $\tilde{\beta} = abde$  resp.) and hence  $\tilde{\alpha}$  and  $\tilde{\beta}$  are adjacent in  $\Lambda(\bar{S})$ .  $\square$

*Remark 4.3.* Let  $T$  be obtained from  $S$  by the edge flip  $ab \mapsto cd$  and  $e, \alpha = abc, \beta = abd, \gamma = acd, \delta = bcd$  as in the proof of Theorem 4.1. Then  $\tilde{\alpha} = abce, \tilde{\delta} = abde, \tilde{\gamma} = acde, \tilde{\delta} = bcde \in \bar{T}$ . Moreover, let the (up to) two nodes adjacent to  $\tilde{\alpha}$  in  $\Lambda(\bar{S})$  be  $acex$  and  $bcey$ , and let the (up to) two nodes adjacent to  $\tilde{\beta}$  in  $\Lambda(\bar{S})$  be  $adez$  and  $bdew$ .

Then the dual graph  $\Lambda(\bar{T})$  is the tree build from  $\Lambda(\bar{S})$ , with set of nodes  $U = \{\sigma \in \bar{S} \mid \sigma \text{ is a tetrahedron}\} \setminus \{\tilde{\alpha}, \tilde{\beta}\} \cup \{\tilde{\gamma}, \tilde{\delta}\}$  with all arcs in  $\Lambda(\bar{S})$  adjacent to  $\tilde{\alpha}$  and  $\tilde{\beta}$  removed, and arcs added between  $\tilde{\gamma}$  and  $\tilde{\delta}$  (corresponding to triangle  $cde$ ),  $\tilde{\gamma}$  and  $acex$  (corresponding to  $ace$ ),  $\tilde{\delta}$  and  $bcey$  ( $bce$ ),  $\tilde{\gamma}$  and  $adez$  ( $ade$ ), and  $\tilde{\delta}$  and  $bdew$  ( $bde$ ), see Figure 4.1.



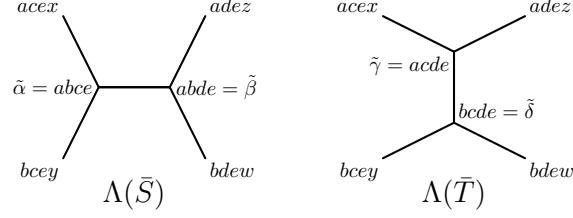


Figure 4.1: Transformation of dual graph by edge flip  $ab \mapsto cd$  in the proof of Theorem 4.1.

For a vertex (node)  $v$  of a graph  $G$ , let the *vertex-deleted subgraph*  $G-v$  be the induced subgraph  $G[V(G) \setminus \{v\}]$ .

**Corollary 4.4.** *Let  $S$  be a stacked 2-sphere,  $\alpha = abc$ ,  $\beta = abd$  two triangles of  $S$ ,  $\tilde{\alpha}$  (resp.,  $\tilde{\beta}$ ) the unique tetrahedron of  $\bar{S}$  containing  $\alpha$  (resp.,  $\beta$ ),  $\sigma \in \bar{S}$  be a degree 4 node in  $\Lambda(\bar{S})$ , and let  $G_1, \dots, G_4$  be the connected components of  $\Lambda(\bar{S}) - \sigma$ . If the 2-sphere  $T$  obtained from  $S$  by the edge flip  $ab \mapsto cd$  is also a stacked 2-sphere then*

- (i)  $\sigma$  is a tetrahedron of  $\bar{T}$ ,
- (ii)  $\sigma$  is a degree 4 node in  $\Lambda(\bar{T})$ ,
- (iii) both  $\tilde{\alpha}$  and  $\tilde{\beta}$  are in one component of  $\Lambda(\bar{S}) - \sigma$ , say in  $G_4$ , and
- (iv) the components of  $\Lambda(\bar{S}) - \sigma$  are  $G_1, G_2, G_3, G'_4$  for some tree  $G'_4$ .

*Proof.* It follows from Theorem 4.1 that  $\text{lk}_{\bar{S}}(ab)$  is a path of the from  $c-e-d$  and  $\tilde{\alpha} = abce$ ,  $\tilde{\beta} = abde$  for some vertex  $e$ . In particular,  $\tilde{\alpha}$  and  $\tilde{\beta}$  are the only two tetrahedra in  $\bar{S}$  containing  $ab$ . Since all the 2-dimensional faces of  $\sigma$  are interior triangles, we have  $\sigma \notin \{\tilde{\alpha}, \tilde{\beta}\}$ . Thus,  $\sigma$  cannot contain the edge  $ab$ . Since  $\sigma$  is a clique in  $\text{Skel}_1(S)$ , this implies that  $\sigma$  is a clique in  $\text{Skel}_1(T)$ . Hence  $\sigma \in \bar{T}$ . This proves part (i).

Observe that  $\{a, c, d, e\}$  and  $\{b, c, d, e\}$  are cliques in  $\text{Skel}_1(T)$ . Therefore,  $\tilde{\gamma} := acde$ ,  $\tilde{\delta} := bcde \in \bar{T}$ . Let  $\tau$  be a 2-dimensional face of  $\sigma$ . Then  $\tau$  is an interior face in  $\bar{S}$ . Let  $\tau = \sigma \cap \mu$  for some tetrahedron  $\mu \in \bar{S}$ . If  $ab \notin \mu$  then  $\mu$  is a clique in  $\text{Skel}_1(T)$  and hence  $\mu \in \bar{T}$ . Then  $\tau = \sigma \cap \mu$  is an interior triangle of  $\bar{T}$ . If  $ab \subset \mu$  then  $\mu$  is  $\tilde{\alpha}$  or  $\tilde{\beta}$ . Assume, without loss, that  $\mu = \tilde{\alpha} = abce$ . Since  $ab \notin \sigma$  and  $\mu \cap \sigma$  is a face of  $\mu$ ,  $\tau = \mu \cap \sigma = ace$  or  $bce$ . Assume, without loss, that  $\tau = ace$ . Then  $\sigma = acex$  for some vertex  $x$  and  $\tau = \sigma \cap \tilde{\gamma}$ . Thus,  $\tau$  is an interior triangle of  $\bar{T}$ . Thus, each 2-dimensional face of  $\sigma$  is an interior triangle of  $\bar{T}$ . Part (ii) follows from this.

Part (iii) follows from the fact that  $\tilde{\alpha}$  and  $\tilde{\beta}$  share a triangle in  $\bar{S}$  which (necessarily) is not a face of  $\sigma$ .

The four 2-dimensional faces of  $\tilde{\gamma} = acde$  are  $acd, ace, ade$  and  $cde$ . Since  $cd$  is a non-edge in  $\bar{S}$ , we have that  $acd, cde$  are not in  $\bar{S}$  and  $acd = \tilde{\gamma} \cap \tilde{\alpha}$ ,  $cde = \tilde{\gamma} \cap \tilde{\beta}$ . Thus, by part (iii),  $\tilde{\gamma}$  is not adjacent to any vertex of  $G_1 \cup G_2 \cup G_3$ . Similarly,  $\tilde{\delta}$  is not adjacent to any vertex of  $G_1 \cup G_2 \cup G_3$ . Part (iv) now follows since the set of nodes of  $\Lambda(\bar{T})$  is  $(\{\tau : \tau \text{ is a tetrahedron in } \bar{S}\} \setminus \{\tilde{\alpha}, \tilde{\beta}\}) \cup \{\tilde{\gamma}, \tilde{\delta}\}$ .  $\square$

**Corollary 4.5.** *Let  $S$  be a stacked 2-sphere,  $T$  a stacked 2-sphere obtained from  $S$  by an edge flip, and let  $V_S$  (resp.,  $V_T$ ) be the set of degree 4 nodes in  $\Lambda(\bar{S})$  (resp., in  $\Lambda(\bar{T})$ ). Then the induced subgraphs  $\Lambda(\bar{S})[V_S]$  and  $\Lambda(\bar{T})[V_T]$  are isomorphic.*

*Proof.* By Corollary 4.4,  $V_S = V_T$ . For  $\sigma_1, \sigma_2 \in V_S = V_T$ ,  $\sigma_1$  and  $\sigma_2$  are adjacent in  $\Lambda(\bar{S})[V_S]$  if and only if  $\sigma_1 \cap \sigma_2$  is an interior triangle of  $\bar{S}$  if and only if  $\sigma_1 \cap \sigma_2$  contains three vertices if and only if  $\sigma_1 \cap \sigma_2$  is an interior triangle of  $\bar{T}$  if and only if  $\sigma_1$  and  $\sigma_2$  are adjacent in  $\Lambda(\bar{T})[V_T]$ . The corollary follows from this observation.  $\square$

**Corollary 4.6.** *The Pachner graph  $\mathcal{S}_n$  of  $n$ -vertex stacked 2-spheres is disconnected for  $n \geq 8$ .*

*Proof.* The stacked 3-ball associated to an  $n$ -vertex stacked 2-sphere,  $n \geq 8$ , has a dual graph with  $m = n - 3 \geq 5$  nodes, and every  $m$  node tree is the dual graph of at least one stacked 3-ball. Hence there exist a stacked 3-ball  $B_1$  with dual graph having one node of degree 4 and  $n - 4$  nodes of degree  $\leq 3$ , and there exist a stacked 3-ball  $B_2$  with dual graph with  $m$  vertices of degree  $\leq 2$ . Then, by Corollary 4.5, the  $n$ -vertex stacked 2-spheres  $\partial B_1$  and  $\partial B_2$  are in different connected components of  $\mathcal{S}_n$ .  $\square$

**Corollary 4.7.** *For  $m \in \mathbb{Z}^+$ , let  $t(m)$  be the number of non-isomorphic  $m$ -vertex trees of maximum degree  $\leq 4$ . Moreover, let  $n = 3m + 5$ . Then the Pachner graph  $\mathcal{S}_n$  of  $n$ -vertex stacked 2-spheres has  $t(m)$  components each containing a single stacked 2-sphere.*

*Proof.* Let  $H$  be a tree on  $m$  nodes of degree  $\leq 4$ . Consider a new graph  $G$  by connecting each node of  $H$  of degree  $i$  to  $(4 - i)$  new nodes. Then  $G$  is a connected acyclic graph and hence a tree. By construction, the number of new nodes in  $G$  equals the number of new arcs in  $G$  which is  $\sum_{v \in V(H)} (4 - \deg_H(v)) = 4m - \sum_{v \in V(H)} \deg_H(v) = 4m - 2(m - 1) = 2m + 2$ . Therefore,  $G$  has  $(m - 1) + (2m + 2) = 3m + 1$  arcs, and thus  $3m + 2$  nodes. It follows that  $G$  has  $m$  nodes of degree 4 and  $2m + 2$  nodes of degree 1, and each degree 1 node of  $G$  is adjacent to a degree four node.

Let  $B$  be a stacked 3-ball whose dual graph  $\Lambda(B)$  is  $G$ . It follows from the definition that we can always construct such a stacked 3-ball. Let  $S = \partial B$ . Since  $S$  is stacked it must have  $3m + 5$  vertices. Let  $\alpha = abc$ ,  $\beta = abd$  be two triangles of  $S$ , and let  $\tilde{\alpha}$  (resp.,  $\tilde{\beta}$ ) be the unique tetrahedron of  $B$  containing  $\alpha$  (resp.,  $\beta$ ). Then  $\deg_{\Lambda(B)}(\tilde{\alpha}), \deg_{\Lambda(B)}(\tilde{\beta}) < 4$  and hence  $\deg_{\Lambda(B)}(\tilde{\alpha}) = 1 = \deg_{\Lambda(B)}(\tilde{\beta})$ . If  $\tilde{\alpha} = \tilde{\beta}$ , then  $cd$  is an edge and hence we cannot perform the edge flip  $ab \mapsto cd$ . If  $\tilde{\alpha} \neq \tilde{\beta}$ , then  $\tilde{\alpha}$  and  $\tilde{\beta}$  are not adjacent in  $\Lambda(B)$  (degree 1 nodes are only adjacent to degree 4 nodes in  $\Lambda(B)$ ) and hence, by Theorem 4.1, the 2-sphere  $T$  obtained from  $S$  by the edge flip  $ab \mapsto cd$  is not stacked. Thus  $S$  is isolated in  $\mathcal{S}_n$ .

If  $H_1$  and  $H_2$  are non-isomorphic trees on  $m$  nodes, then the above construction carried out for both  $H_1$  and  $H_2$  leads to two non-isomorphic trees  $G_1$  and  $G_2$ , leading to two non-isomorphic stacked 3-balls  $B_1$  and  $B_2$  with, by Lemma 2.6, non-isomorphic boundaries  $S_1$  and  $S_2$ . Since there exist at least  $t(m)$  non-isomorphic  $m$  vertex trees of maximum degree  $\leq 4$ , we have at least  $t(m)$  singleton components in  $\mathcal{S}_n$ .  $\square$

**Corollary 4.8.** *The number of connected components in  $\mathcal{S}_n$  is bounded from below by  $C^n$ , for some real number  $C > 1$ .*

*Proof.* Let  $m = \lfloor \frac{n-5}{3} \rfloor$ . Let  $t(m)$  be the number of non-isomorphic  $m$ -vertex trees of maximum degree  $\leq 4$  as in Corollary 4.7.

**Claim:** The number of components in  $\mathcal{S}_n$  is at least  $t(m)$ .

Let  $\mathcal{T}$  be the set of all trees with  $m$  nodes of maximum degree  $\leq 4$ . For each  $H \in \mathcal{T}$ , we can construct a  $(3m + 5)$ -nodes tree  $G$  whose degree 4 nodes are the nodes of  $H$  and all others are of degree 1. By adding  $n - 3m - 5 (\leq 2)$  new nodes to the  $n - 3m - 2$  degree 1 nodes of  $G$  we obtain a new tree  $G'$  having the same set of degree 4 nodes as in  $G$ . Let  $B$  be a stacked 3-ball whose dual graph is  $G'$  and let  $S = \partial B$ . By construction,  $S$  is a stacked 2-sphere with exactly  $n$  vertices. Let  $V_S$  be as in Corollary 4.5. Then  $G'[V_S] = G[V_S] = H$ . Therefore, by Corollary 4.5, the  $n$ -vertex stacked 2-spheres obtained in this process corresponding to different graphs in  $\mathcal{T}$  are in different components of  $\mathcal{S}_n$ . This proves the claim.

Since  $t(m)$  is exponential in  $m$ , the result follows from the claim.  $\square$

Following arguments along the lines of Corollary 4.4 we can observe that, apart from a large number of isolated singleton components in  $\mathcal{S}_n$ , there are also larger connected components corresponding to dual graphs with no, or very few nodes of degree 4. For instance, the largest connected component in  $\mathcal{S}_n$ ,  $n \leq 14$ , shown in Table 1, corresponds to boundaries  $S$  of stacked balls  $\bar{S}$  with dual graphs without degree 4-vertices (i.e.,  $V_S = \emptyset$ ). Let  $\mathcal{S}_n^0$  denote the Pachner graph consisting of this class of stacked 2-spheres. We have the following result.

**Theorem 4.9.** *The Pachner graph  $\mathcal{S}_n^0$  is connected.*

We split the proof of Theorem 4.9 into two lemmas.

**Lemma 4.10.** *Each stacked 2-sphere  $S \in \mathcal{S}_n^0$  is connected to a stacked 2-sphere  $T$  in the Pachner graph  $\mathcal{S}_n^0$ , where the dual graph  $\Lambda(\overline{T})$  of  $\overline{T}$  is a path.*

*Proof.* The idea of the proof is to show that, for every  $S \in \mathcal{S}_n^0$  with  $\Lambda(\overline{S})$  not a path,  $S$  is connected in  $\mathcal{S}_n^0$  to a stacked 2-sphere  $T \in \mathcal{S}_n^0$  with the number of nodes of degree three in  $\Lambda(\overline{T})$  less than that in  $\Lambda(\overline{S})$ .

For  $S \in \mathcal{S}_n^0$  and  $\alpha, \beta$  nodes in  $\Lambda(\overline{S})$ , let  $d_S(\alpha, \beta)$  be the length of the unique path from  $\alpha$  to  $\beta$  in the tree  $\Lambda(\overline{S})$ . Moreover, if  $S$  has a degree three node in  $\Lambda(\overline{S})$ , let  $\ell(S) = \min\{d_S(\alpha, \beta) \mid \alpha \text{ leaf, } \beta \text{ degree three in } \Lambda(\overline{S})\}$ .

**Claim 1:** Let  $S \in \mathcal{S}_n^0$  be a stacked 2-sphere such that  $\Lambda(\overline{S})$  is not a path. If  $\ell(S) \geq 2$  then there exists a stacked 2-sphere  $T \in \mathcal{S}_n^0$  such that (i)  $S$  is connected to  $T$  in  $\mathcal{S}_n^0$ , (ii) the number of degree three nodes in  $\Lambda(\overline{T})$  is the same as in  $\Lambda(\overline{S})$  and (iii)  $\ell(T) = \ell(S) - 1$ .

Let  $\ell = \ell(S) = d_S(\gamma, \delta)$ , where  $\gamma$  is a degree three node and  $\delta$  is a leaf in  $\Lambda(\overline{S})$ . Let  $\gamma = \gamma_0 - \gamma_1 - \dots - \gamma_\ell = \delta$  be the path in  $\Lambda(\overline{S})$  from  $\gamma$  to  $\delta$ . Then  $\deg_{\Lambda(\overline{S})}(\gamma_0) = 3$ ,  $\deg_{\Lambda(\overline{S})}(\gamma_i) = 2$  for  $1 \leq i \leq \ell - 1$ , and  $\deg_{\Lambda(\overline{S})}(\gamma_\ell) = 1$ . Let the other nodes adjacent to  $\gamma$  be  $\alpha$  and  $\beta$ . Assume, without loss, that  $\gamma = 1234$ ,  $\alpha = 124a$ ,  $\beta = 134b$  and  $\gamma_1 = 123x_1$ . Then, the link of 23 in  $\overline{S}$  is of the form  $4-1-x_1-\dots-x_k$  for some  $k \leq \ell$ .

**Case 1.** Let  $k = 1$ . It follows that  $23x_1$  is a triangle of  $S = \partial\overline{S}$ . By Theorem 4.1, the triangulated 2-sphere  $T$  obtained from  $S$  by the edge flip  $23 \mapsto 4x_1$  is stacked and hence, by Corollary 4.4, is in  $\mathcal{S}_n^0$ . By Lemma 2.6,  $\gamma' := 134x_1$  and  $\gamma'_1 := 124x_1$  are tetrahedra in  $\overline{T}$ .

Following the transformation of the dual graph of a stacked ball under an edge flip, as shown in Figure 4.1, the dual graph  $\Lambda(\overline{T})$  is obtained from  $\Lambda(\overline{S})$  by replacing the three edges adjacent to  $\gamma$  with the path  $\beta - \gamma' - \gamma'_1 - \alpha$ , and attaching the path  $\gamma_2 - \dots - \gamma_\ell$  to either  $\gamma'$  or  $\gamma'_1$ . In either case, the path from the new degree three node to  $\gamma_\ell$  is of length  $\ell - 1$ , and since the remaining part of  $\Lambda(\overline{T})$  is equal to the remaining part of  $\Lambda(\overline{S})$ , we have  $\ell(T) = \ell(S) - 1$  and Claim 1 is true in this case.

**Case 2.** Let  $k \geq 2$ . In this case we can assume that  $\gamma_i = 23x_{i-1}x_i$  for  $2 \leq i \leq k$ , and that the triangles  $21x_1, 2x_1x_2, \dots, 2x_{k-2}x_{k-1}, 31x_1, 3x_1x_2, \dots, 3x_{k-2}x_{k-1} \in S$  (i.e., are in the boundary). Since  $\deg_{\Lambda(\overline{S})}(\gamma_k) \leq 2$  ( $= 1$  if  $k = \ell$  and  $= 2$  if  $k < \ell$ ), at least two 2-dimensional faces of  $\gamma_k$  are triangles of  $S$ . This implies that at least one of the triangles  $2x_{k-1}x_k$  and  $3x_{k-1}x_k$  is a triangle of  $S$ .

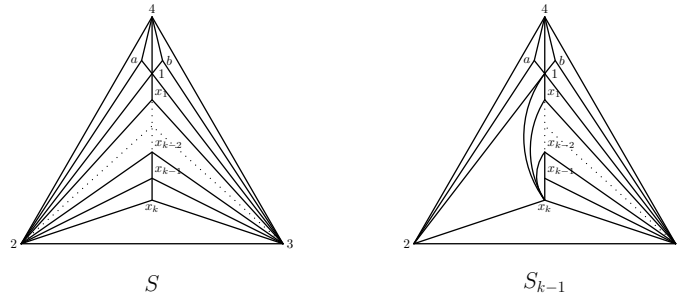


Figure 4.2: Sequence of edge flips as performed in the proof of Lemma 4.10, Claim 1, Case 2.

Assume, without loss, that  $2x_{k-1}x_k \in S$ . Let  $S_1$  be obtained from  $S = S_0$  by the edge flip  $2x_{k-1} \mapsto x_kx_{k-2}$ . Since  $\text{lk}_{\overline{S}}(2x_{k-1}) = x_{k-2}-3-x_k$ , by Theorem 4.1,  $S_1$  is stacked. Observe that the path  $\gamma_{k-2}-\gamma_{k-1}-\gamma_k-\gamma_{k+1}$  in  $\Lambda(\overline{S})$  is replaced by  $\gamma_{k-2}-(23x_kx_{k-2})-(3x_kx_{k-1}x_{k-2})-\gamma_{k+1}$  in  $\Lambda(\overline{S}_1)$ . Thus,  $\Lambda(\overline{S}_1)$  is isomorphic to  $\Lambda(\overline{S})$ .

Inductively, for  $1 \leq i \leq k - 1$ ,  $\text{lk}_{\overline{S}}(2x_{k-i}) = x_{k-i-1}-3-x_k$  and hence the sphere  $S_i$  obtained from  $S_{i-1}$  by the edge flip  $2x_{k-i} \mapsto x_kx_{k-i-1}$  is stacked. Then  $\Lambda(\overline{S}_i)$  is isomorphic to  $\Lambda(\overline{S}_{i-1})$ , see

Figure 4.2. (Note that  $S_{k-1}$  is obtained by the sequence of edge flips  $2x_{k-1} \mapsto x_k x_{k-2}$ ,  $2x_{k-2} \mapsto x_k x_{k-3}, \dots, 2x_2 \mapsto x_k x_1$ ,  $2x_1 \mapsto x_k 1$ .)

It follows that  $S_{k-1}$  is stacked,  $S$  can be joined to  $S_{k-1}$  in  $\mathcal{S}_n^0$ ,  $\Lambda(\overline{S}_{k-1})$  is isomorphic to  $\Lambda(\overline{S})$ , and  $\text{lk}_{\overline{S}_{k-1}}(23) = 4-1-x_k$ . In particular,  $S_{k-1}$  satisfies the hypothesis of Case 1,  $\ell(S_{k-1}) = \ell(S)$  and the number of degree three nodes in  $\Lambda(\overline{S}_{k-1})$  is the same as that in  $\Lambda(\overline{S})$ . Consequently, by Case 1,  $S_{k-1}$  is connected to some  $T$  in  $\mathcal{S}_n^0$ , such that the number of degree three nodes in  $\Lambda(\overline{T})$  is the same as that in  $\Lambda(\overline{S}_{k-1})$  (which is the same as that in  $\Lambda(\overline{S})$ ) and  $\ell(T) = \ell(S_{k-1}) - 1 = \ell(S) - 1$ . This completes the proof of Claim 1.

**Claim 2:** For  $S \in \mathcal{S}_n^0$ , if  $\Lambda(\overline{S})$  has a leaf which is adjacent to a degree 3 node in  $\Lambda(\overline{S})$  (i.e.,  $\ell(S) = 1$ ) then there exists  $T \in \mathcal{S}_n^0$  which can be obtained from  $S$  by an edge flip and the number of nodes of degree 3 in  $\Lambda(\overline{T})$  is one less than that in  $\Lambda(\overline{S})$ .

Let  $\delta = 123d$  be a leaf node which is adjacent to a degree three node  $\gamma = 1234$ . Assume, as above, that the adjacent nodes of  $\gamma$  are  $\alpha = 124a$  and  $\beta = 134b$ . Then edge  $23$  is in two tetrahedra and, by Theorem 4.1, the 2-sphere  $T$  obtained from  $S$  by the edge flip  $23 \mapsto 4d$  is stacked and hence in  $\mathcal{S}_n^0$  by Corollary 4.4. Moreover, by Proposition 2.6,  $\alpha' := 124d$  and  $\gamma' := 134d$  are in  $\overline{T}$ . Again, by following the transformation shown in Figure 4.1,  $\Lambda(\overline{T})$  contains the path  $\delta-\gamma'-\alpha'-\beta$  instead of the three edges adjacent to  $\gamma$  in  $\Lambda(\overline{S})$ . Since the remaining parts of  $\Lambda(\overline{S})$  and  $\Lambda(\overline{T})$  coincide, Claim 2 follows.

The result follows inductively using Claims 1 and 2.  $\square$

**Lemma 4.11.** Let  $\partial\Delta_n$  be as shown in Figure 4.4 and let  $S \in \mathcal{S}_n^0$ . If  $\Lambda(\overline{S})$  is a path then  $S$  is connected to  $\partial\Delta_n$  in  $\mathcal{S}_n^0$ .

*Proof.* Let  $\Lambda(\overline{S}) = \gamma_1-\gamma_2-\dots-\gamma_{n-3}$ .

**Claim:** If  $\gamma_1, \dots, \gamma_k$  have a common edge and  $\gamma_1, \dots, \gamma_{k+1}$  have no common edge,  $k \leq n-4$ , then  $S$  can be joined to  $T \in \mathcal{S}_n^0$ , where  $\Lambda(\overline{T})$  is a path of the form  $\alpha_1-\alpha_2-\dots-\alpha_{n-3}$  such that  $\alpha_1, \dots, \alpha_{k+1}$  have a common edge.

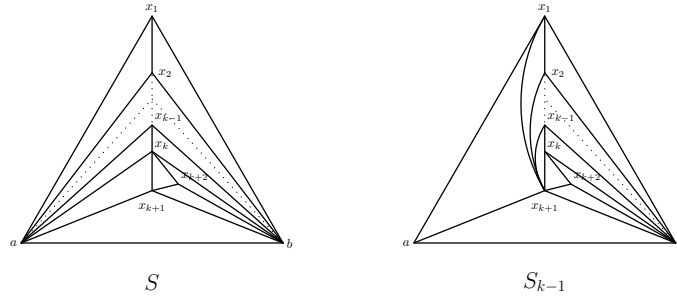


Figure 4.3: Sequence of edge flips as performed in the proof of Lemma 4.11.

Since  $\gamma_1, \dots, \gamma_{k+1}$  have no common edge, we can assume that  $k \geq 3$ . Let  $\gamma_i = abx_i x_{i+1}$  for  $1 \leq i \leq k$ . Assume without loss of generality that  $\gamma_{k+1} = bx_k x_{k+1} x_{k+2}$ . Then  $\text{lk}_{\overline{T}}(ax_k) = x_{k-1}-b-x_{k+1}$ . Thus, by Theorem 4.1, the 2-sphere  $S_1$  obtained from  $S$  by the edge flip  $ax_k \mapsto x_{k+1}x_{k-1}$  is stacked. Similarly, the 2-sphere  $S_2$  obtained from  $S_1$  by the edge flip  $ax_{k-1} \mapsto x_{k+1}x_{k-2}$  is stacked. Continuing this way, we obtain a stacked sphere  $T = S_{k-1}$  from  $S_{k-2}$  by the edge flip  $ax_2 \mapsto x_{k+1}x_1$ , see Figure 4.3. Hence  $S$  can be joined to  $T$  in  $\mathcal{S}_n^0$  and  $\Lambda(\overline{T}) = \alpha_1-\alpha_2-\dots-\alpha_{k+1}-\gamma_{k+2}-\dots-\gamma_{n-3}$ , where  $\alpha_{k+1} = bx_{k+1}x_{k+2}x_k$ ,  $\alpha_i = bx_{k+1}x_i x_{i-1}$ ,  $2 \leq i \leq k$ , and  $\alpha_1 = bx_{k+1}x_1 a$ . This proves the claim.

The lemma follows by induction using the claim.  $\square$

*Proof 1 of Theorem 4.9.* The result follows from Lemmas 4.10 and 4.11.  $\square$

*Proof 2 of Theorem 4.9.* Consider the  $n$ -vertex stacked 3-ball  $\Delta_n$  whose tetrahedra all share one edge. Its dual graph is a path of length  $n - 4$  and the tetrahedra are of the form  $1234, 2345, 2356, \dots, 23(n-1)n$ , see Figure 4.4. We have  $\partial\Delta_n \in \mathcal{S}_n^0$ . The idea of the proof is to connect an arbitrary  $n$ -vertex stacked 2-sphere  $S \in \mathcal{S}_n^0$  to  $\partial\Delta_n$  by a sequence of edge flips in  $\mathcal{S}_n^0$ .

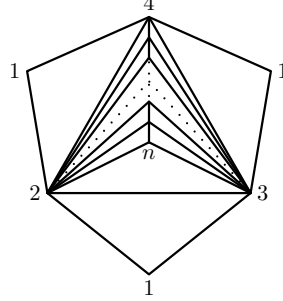


Figure 4.4: The canonical stacked 3-ball  $\Delta_n$ . Note that this complex is also used as a canonical target in [3] to prove upper bounds on the diameter of the Pachner graph  $\mathcal{P}_n$  of  $n$ -vertex 2-spheres.

Let  $S \in \mathcal{S}_n^0$ . By Theorem 4.1, each edge flip across an edge of  $S$  which is contained in only two (adjacent) tetrahedra of  $\bar{S}$  yields a new stacked 2-sphere  $T$ . In this case, we automatically have  $T \in \mathcal{S}_n^0$  by Corollary 4.5. Hence, to ensure we remain within  $\mathcal{S}_n^0$  when performing an edge flip, it is enough to check that we remain within the Pachner graph  $\mathcal{S}_n$ .

**Terminology:** Choose an arbitrary node  $r$  of  $\Lambda(\bar{S})$  as *root node*. For a node  $\delta \in \Lambda(\bar{S})$ , a node  $\gamma$  adjacent to  $\delta$  is called a *child node* of  $\delta$ , if the unique path from  $r$  to  $\gamma$  contains  $\delta$ , otherwise  $\gamma$  is called a *parent node* of  $\delta$ . A *leaf node* is a node without child nodes. A *subtree* at  $\delta \in \Lambda(\bar{S})$  is the tree containing  $\delta$  as well as all nodes underneath it.

**Outline:** We describe three procedures to be applied to subtrees at nodes  $\delta \in \Lambda(\bar{S})$  of three very special structures. This is followed by a description of how to transform an arbitrary stacked 2-sphere  $S \in \mathcal{S}_n^0$  into  $\Delta_n$  staying within  $\mathcal{S}_n^0$  and using nothing but these three procedures.

For the remainder of this proof and unless otherwise stated,  $\delta = abcd$  is a node in  $\Lambda(\bar{S})$  such that the parent node (if existent) is attached to  $abc$ , i.e. all child nodes contain  $d$ .

**Procedure 1:** The node  $\delta$  is a leaf node. In this case we do nothing.

**Procedure 2:** The stacked 3-ball  $B$  corresponding to the subtree containing  $\delta$  and all of its nodes underneath is isomorphic to  $\Delta_k$ ,  $k \geq 5$  (note that, if  $k = 4$ ,  $\delta$  must be a leaf node), and at least one edge of  $ab$ ,  $ac$ , or  $bc$  contains all  $k - 3$  tetrahedra. Procedure 2 consists of changing this edge into any of the other two.

After relabeling, let the unique child node of  $\delta$  be attached to  $bcd$ , and let  $bc$  be the edge contained in all  $k - 3$  tetrahedra, see Figure 4.5. Moreover, assume that we want  $ab$  to be the edge contained in all  $k - 3$  tetrahedra of the subtree after applying the procedure. In this case, Procedure 2 consists of  $k - 5$  edge flips on  $B$  as indicated in Figure 4.5. This yields a subcomplex associated to the subtree, still isomorphic to  $\Delta_k$ , with the desired properties. All other cases can be obtained by relabeling or applying the reverse procedure.

It follows that, whenever the complex associated to a subtree starting at a tetrahedron  $\delta$  is isomorphic to  $\Delta_k$  with one of the edges  $ab$ ,  $ac$ , or  $bd$  containing all  $k - 3$  tetrahedra, Procedure 2 enables us to choose which of these edges is contained in all tetrahedra.

**Procedure 3:** The subtree containing  $\delta$  has two child nodes  $\gamma_1$  and  $\gamma_2$  with subtrees isomorphic to  $\Delta_j$  and  $\Delta_\ell$ , for some  $j, \ell \geq 4$ , such that the edges containing all of their tetrahedra is one of the edges of  $\delta$ . Procedure 3 turns the subtree at  $\delta$  into a complex isomorphic to  $\Delta_{j+\ell-2}$ .

Assume that, possibly after relabeling, the child tetrahedra are attached to  $adc$  and  $bcd$ . After applying Procedure 2 to the complexes associated to the subtrees at  $\gamma_1$  and  $\gamma_2$  we can assume that

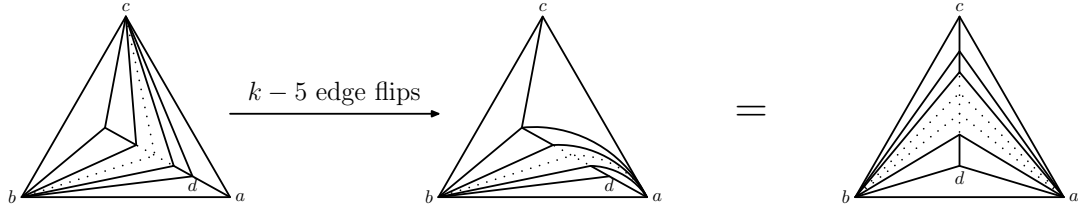


Figure 4.5: Procedure 2 in the case the unique child node is attached to  $bcd$  and edge  $bc$  is contained in all tetrahedra of the subtree complex.

edges  $bc$  and  $ac$  contain all tetrahedra of the respective subtrees at  $\gamma_1$  and  $\gamma_2$ , see Figure 4.6. We perform a sequence of edge flips as indicated in Figure 4.6 to obtain a single subtree attached to  $\delta$  isomorphic to  $\Delta_{j+\ell-2}$ .

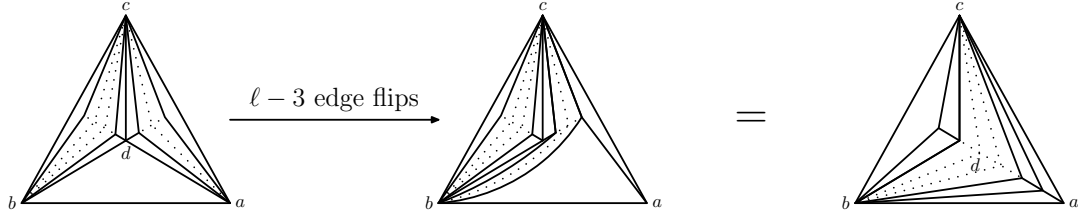


Figure 4.6: Procedure 3: merging two subtrees isomorphic to  $\Delta_j$  and  $\Delta_\ell$  into one subtree isomorphic to  $\Delta_{j+\ell-2}$ .

**Transformation:** The result now follows by the following argument. Choose a root node of degree  $\leq 2$ . We start by moving up from the leaves, where nothing needs to be done, see Procedure 1. Whenever a node  $\delta$  has one child node  $\gamma$ , we know from the structure of the transformation that the subtree at  $\gamma$  is isomorphic to  $\Delta_k$ ,  $k \geq 4$ , where at least one edge containing all of its tetrahedra must be an edge of  $\delta$  (and hence the subtree at  $\delta$  must be isomorphic to  $\Delta_{k+1}$ ). We apply Procedure 2 to the subtree at  $\gamma$  until this edge is one of  $ab$ ,  $ac$ , or  $bd$ . (Note that applying Procedure 2 to the subtree at  $\gamma$  justifies the assumption on the subtree of  $\gamma$  made above.) If  $\delta$  has two child nodes  $\gamma_1$  and  $\gamma_2$  we can assume from the previous steps that both must be isomorphic to  $\Delta_j$  and  $\Delta_\ell$  for some  $j, \ell \geq 4$  with the respective edges contained in all tetrahedra of their subcomplexes being edges of  $\delta$ , and hence we can apply Procedure 3 to  $\delta$ .

Note that, by assumption and by the choice of the root,  $\delta$  must have  $\leq 2$  children.

Repeating this process iteratively from the leaves up yields an  $n$ -vertex stacked 3-ball isomorphic to  $\Delta_n$ . Note that each edge flip in both flip sequences shown in Figures 4.5 and 4.6 goes across an edge of degree 2 in the corresponding stacked 3-ball and thus all intermediate 2-spheres are stacked 2-spheres in  $\mathcal{S}_n^0$ .  $\square$

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